Dissertation

# The Generalised Symmetric Group: Transitivity, Designs and Perfect Matchings

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*"Life is a lesson that we don't sign up for"*

— After The Burial, *Neo Seoul*

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# ABSTRACT

The central topic of this thesis is the study of codes and designs in several association schemes related to the generalised symmetric group. We investigate the conjugacy class scheme of the generalised symmetric group (which includes Coxeter groups and the symmetry groups of regular polytopes) and the perfect matching association scheme.

In the conjugacy class scheme, we find that groups that are transitive on coloured numbers or regular polytopes are designs, and in the perfect matching association scheme, we find that 1-factorisations and hyperfactorisations are examples of designs. We generalise these notions and set them in the context of association schemes, unifying and vastly generalising existing results. This allows us to derive divisibility conditions on the parameters and new existence and non-existence results.

In both schemes, we develop a characterisation of the subsets of interest as Delsarte designs. Our methods use representation and character theory, especially the decomposition of permutation characters. Using our characterisation of designs, we prove generalisations of the celebrated Livingstone–Wagner theorem about $t$-homogeneity in the symmetric group. We also give constructions and an application of our results to the famous open problem of the existence of a finite projective plane whose order is not a prime power.

# ZUSAMMENFASSUNG

Das zentrale Thema dieser Dissertation ist die Erforschung von Codes und Designs in verschiedenen Assoziationsschemata, die zur verallgemeinerten symmetrischen Gruppe gehören. Wir untersuchen das Konjugationsklassenschema der verallgemeinerten symmetrischen Gruppe (Spezialfälle davon sind Coxetergruppen und die Symmetriegruppen regulärer Polytope) und das Assoziationsschema der perfekten Matchings.

Im Konjugationsklassenschema zeigen wir, dass Gruppen, die transitiv auf gefärbten Zahlen oder regulären Polytopen operieren, Designs sind, und im Assoziationsschema der perfekten Matchings zeigen wir, dass 1-Faktorisierungen und Hyperfaktorisierungen Beispiele für Designs sind. Wir verallgemeinern diese Strukturen und ordnen sie in die Theorie der Assoziationsschemata ein, wodurch wir bereits existierende Resultate vereinen und deutlich verallgemeinern. Dadurch können wir Teilbarkeitsbedingungen an die Parameter sowie neue Existenz- und Nichtexistenzresultate herleiten.

In beiden Assoziationsschemata entwickeln wir eine Charakterisierung der untersuchten Teilmengen als Delsarte-Designs. Wir verwenden Darstellungs- und Charaktertheorie, insbesondere die Zerlegung von Permutationscharakteren. Mit Hilfe unserer Charakterisierung der Teilmengen als Designs beweisen wir Verallgemeinerungen des Livingstone–Wagner-Theorems über $t$-Homogenität in der symmetrischen Gruppe. Außerdem geben wir explizite Konstruktionen an und wenden unsere Ergebnisse auf das bekannte ungelöste Problem der Existenz einer endlichen projektiven Ebene, deren Ordnung keine Primzahlpotenz ist, an.

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# 1 INTRODUCTION

In this thesis, we study codes and designs in various association schemes. The connecting element between them is the *hyperoctahedral group* $C_2 \wr S_n$. We show how group actions and methods from representation theory and finite geometry work together to produce results about codes and designs. The designs that we investigate will turn out to have a connection to transitive group actions and factorisations of graphs using perfect matchings.

In general, one can think of a 'code' in an association scheme as a set of elements that are in a certain sense 'far apart'. One of the most important applications is the theory of error-correcting codes: How can messages be encoded so that they can still be decoded correctly even if errors are introduced during transmission? Here, we want the messages to be far apart (in some kind of metric) so that a few errors cannot turn an element of a code into another one. This is the foundation of all of our modern forms of digital communication, including messaging, streaming of videos and file sharing.

On the other hand, a 'design' in an association scheme can be thought of as a subset that 'approximates' the whole set. This can often be phrased in terms of properties like 'a constant number of occurrences' or 'evenly covering a certain set of objects'. A classical example is given by the *Fano plane* (see Figure 1.1).

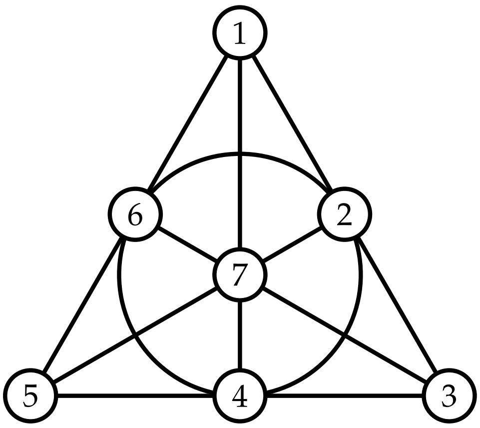



Figure 1.1: The Fano plane

It is a collection of seven subsets of the numbers 1 to 7, each containing exactly 3 elements. Every pair of distinct numbers is contained in a constant number of subsets (in fact, for the Fano plane it is precisely 1 subset). This property of a constant number of occurrences is clearly fulfilled by the set of *all* subsets of size 3. However, as the Fano plane shows, it is possible to use less. In that sense, a design approximates the whole set: The 7 subsets of size 3 cover all 21 (unordered) pairs of distinct elements evenly, just like the whole set of subsets of size 3 does.

Let us give a formal definition of block designs.

**Definition 1.1.1.** Let $t, v, k \in \mathbb{N}$ with $t \leqslant k \leqslant v$ and let $X$ be a finite set with $|X| = v$. A set $D$ of subsets of $X$ is called a *$t$-$(v, k, \lambda)$ (block) design* (or *$t$-design* for short) if every subset $B \in D$ has precisely $k$ elements and for every subset $T \subseteq X$ of size $t$ there are precisely $\lambda > 0$ subsets $B \in D$ such that $T \subseteq B$. The subsets $B \in D$ are called *blocks*.

Notice that this definition is in spirit with our intuition that a design in an association scheme represents an 'even covering'. The blocks of a $t$-$(v, k, \lambda)$ design cover every subset of size $t$ constantly often. Note that this definition does not allow repeated blocks. We could allow this and then call block designs without repeated blocks *simple* designs. We will later see that block designs can be interpreted in terms of association schemes.

Block designs with parameters 2-$(v, 3, 1)$ are called *Steiner triple systems*. The Fano plane from Figure 1.1 is such a Steiner triple system. They are named after Jakob Steiner who introduced them in 1853 [59]. He proved necessary conditions on the parameter $v$ such that a Steiner triple system can exist and asked whether these conditions are also sufficient. However, Thomas Kirkman had already solved the problem in 1847 [34], showing that Steiner triple systems exist for all $v$ with $v \equiv 1$ or 3 mod 6. Thus, it seems that Steiner was unaware of Kirkman's results. As Steiner's paper received more attention from the mathematical community, the triple systems were named after him. Nowadays, the term *Steiner system* refers more generally to any $t$-design with $\lambda = 1$.

The theory of block designs has applications in the theory of design of experiments in statistics (see for example [2]). Roughly speaking, this is about how to conduct statistical experiments in order to establish desired properties like validity and reliability. Sample applications are comparing different kinds of medicine, or software testing where every combination of inputs has to be checked.

The theory of design of statistical experiments has given rise to the theory of *association schemes* (see [2, Chapter 13] for a short historic overview). Association schemes have proven themselves to be a useful tool in many areas of algebraic combinatorics. They can be thought of as a framework to phrase all of the different instances of codes and designs that we talk about in this introduction in a common language and they provide powerful techniques to investigate them.

These techniques are largely due to Philippe Delsarte. In his seminal thesis [19], he investigated the *Hamming scheme* and the *Johnson scheme*, showing that they provide a unifying framework to study error-correcting codes and block designs. This demonstrated the strength and significance of association schemes in combinatorics. He generalised these notions of codes and designs to so called *D-cliques* and *T-designs* and provided a rich theory to study them. Motivated by the classical examples of error-correcting codes (in the Hamming scheme) and combinatorial block designs (in the Johnson scheme), the concepts of 'codes' and 'designs' have since been investigated in many other contexts.

The definition of a $T$-design in an association scheme is purely algebraic. For a subset $Y$, one can compute a sequence of non-negative numbers called the *dual distribution*. Then, a subset $Y$ is a $T$-design if its dual distribution has zeroes in certain places that are specified by $T$. Precise definitions and examples are given in Section 3.2.

In general, the definition of $T$-designs raises the following questions.

(1) If $Y$ is a $T$-design in an association scheme, then what are its combinatorial properties?

(2) If $Y$ is a set with a 'nice' combinatorial property, is it a $T$-design in some association scheme?

While the definition of a $T$-design is not inherently combinatorial, it gives rise to structures with interesting combinatorial properties in a vast number of situations. See for example [47, Section 8] for some commutative schemes, [58] for classical schemes and [20] for an approach to understanding $T$-designs using the idea of 'covering' in semilattices. This motivates the first question. On the other hand, being able to phrase an existing problem in the language of association schemes can give new insights and give rise to new techniques to study the problem, motivating the second question.

In particular, the interpretation of a combinatorial problem in terms of $T$-designs is very helpful when one wants to compare designs. That is, if we have a combinatorial property $A$ and a combinatorial property $B$, we can ask whether every subset with property $A$ also has property $B$. It turns out that while this question can be hard to answer directly, it can become easy to answer once we can phrase the combinatorial properties in the language of $T$-designs. We will see examples of this in Sections 5.5 and 6.3.

In this thesis, we answer these questions for some association schemes connected to the hyperoctahedral group. We investigate the *conjugacy class* *scheme* of the wreath product $G \wr S_n$ (which includes the group $C_r \wr S_n$, the *generalised symmetric group*) and the *perfect matching association scheme* and study their codes and designs. Both are connected to well-studied and important objects.

## DESIGNS IN THE GENERALISED SYMMETRIC GROUP

Codes and designs in the symmetric group have been studied extensively. *Permutation codes* are a special case of codes in the association scheme of the symmetric group and they have many practical applications, for example in powerline communication [15]. On the other hand, *transitive subgroups* are a special case of designs in the symmetric group and the concept of transitive group actions has ubiquitous applications, both mathematical and practical.

In [46], Martin and Sagan define a new notion of transitivity for subsets of permutations of the symmetric group. They show that subsets with such a transitivity property are designs in the conjugacy class scheme of the symmetric group. As the symmetric group is a Coxeter group of type $A_n$, they ask whether similar results hold for the Coxeter group of type $B_n$, which is the *hyperoctahedral group* $C_2 \wr S_n$. We answer this question affirmatively by investigating designs in the conjugacy class scheme of the wreath product $G \wr S_n$ where $G$ is a finite abelian group. The case $G = C_2$ gives the hyperoctahedral group. We also give an application of our results to the long-standing and still unproven *prime power conjecture* for finite projective planes.

For $n \in \mathbb{N}$ and the cyclic group $C_r$ of order $r$, the group $C_r \wr S_n$ is called the *generalised symmetric group*. It can be thought of as permutations of $n$ objects (that are permuted by $S_n$) where additionally each object has one of $r$ different colours that are permuted by the group $C_r$. We give will a more detailed description of this group later. For now, we will focus on the case $G = C_2$ and give a visual intuition of what designs in this group look like.

The group $C_2 \wr S_n$ is the symmetry group of a *hypercube*, i.e. a higher-dimensional analogue of the classical 3-dimensional cube. Equivalently, as the cube is dual to the octahedron, it is also the symmetry group of a *hyperoctahedron*. Hence, it is also called the hyperoctahedral group.

The group $C_2 \wr S_3$ is the symmetry group of the classical 3-dimensional cube which has 6 faces, 12 edges and 8 vertices. We have $|C_2 \wr S_3| = 48$, that is a cube has precisely 48 symmetries. They are compositions of rotations and reflections. Furthermore, out of these 48 symmetries, precisely 24 are rotations and they form the rotation group of the cube.

In order to give an example of a design, we will think about how the symmetry group of the cube acts on its substructures like faces and vertices.

The rotations of the cube induce permutations of its 6 faces. It is clear that the rotations act transitively on the faces of the cube as anyone who has played board games involving dice knows – by rotating a die, one can make any number on the die move to any side of the die. Thus, it follows that the rotations act very ‘evenly’ on the cube: Given any two (not necessarily distinct) faces of the cube, there is a constant number of rotations that move the first face to the second. This constant does not depend on the specific choice of the faces. It is the size of a stabiliser of a face and all of them have the same size. In our example, this constant is 4 as there are 4 rotations stabilising a given face (the constant would be 8 if we considered the full symmetry group $C_2 \wr S_3$).

However, we can do better: There is a group $G$ consisting of 6 rotations of the cube that also acts transitively on the faces. Here, for any pair of faces, exactly one of the 6 rotations maps the first face to the second. This group of rotations is isomorphic to $S_3$ and its generators are given in Figure 1.2. The first rotation fixes two opposite vertices and has order 3. The second rotation fixes the midpoints of two opposite edges and has order 2.

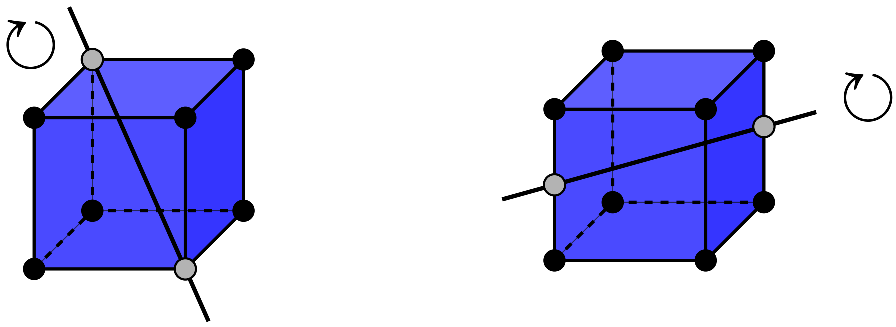


Figure 1.2: Two rotations of the cube

The first rotation permutes each of the three faces adjacent to the grey vertices amongst themselves. The second rotation exchanges these two sets of three faces. Thus, we see that the group that these rotations generate acts transitively on the faces of the cube.

Recalling the idea from before that a design ‘approximates’ the whole set, we see how this manifests in the symmetry group of the cube. Our subgroup $G$ acts on the faces just as evenly as the whole group does. In fact, we will see later (see Example 5.6.9) that the group $G$ is an example of a 1-design in the conjugacy class scheme of the symmetry group of the cube.

We can generalise this example to other substructures of the cube. Groups that act transitively on the vertices or edges of the cube are also examples of designs. We can also go further and consider combinations like transitivity on a face with a specified vertex inside or a face with a specified edge inside. Examples of different transitivity types are given in Figure 1.3.

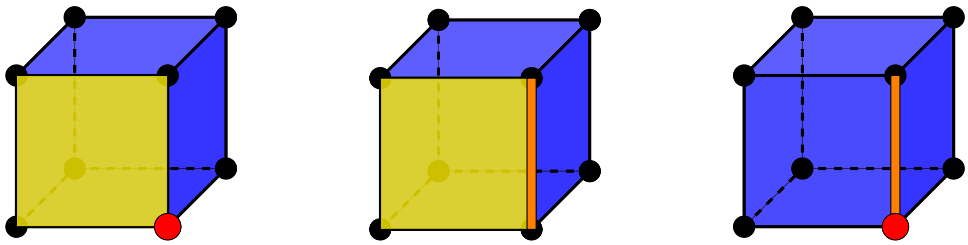


Figure 1.3: Some substructures of the cube that a group can act transitively on.

There are exactly 24 substructures of each of the types in Figure 1.3. There are 6 faces, each with 4 vertices and 4 edges inside. Similarly, there are 12 edges, each with 2 vertices inside. Here, the number of substructures of these types turn out to be the same. In general, the counts can be quite different.

The idea of transitivity on substructures of the cube easily generalises to the hypercube in higher dimensions. Here, we can consider so called *flags* of the hypercube. These are chains of $k$-dimensional faces, for example a 2-dimensional face that is contained in a 4-dimensional face which is in turn contained in a 5-dimensional face. Then, we can look for groups that act transitively on the set of all flags of some type. Every such group is an example of a design in the group $C_2 \wr S_n$. Note that the examples from Figure 1.3 can be phrased in the language of flags.

Viewing the substructures of the hypercube as chains of $k$-dimensional faces, we see that this idea can be generalised to any *regular polytope*. These are polytopes whose symmetry group acts transitively on the 'complete' flags, i.e. chains $F_0 \subseteq F_1 \subseteq F_2 \dots$ where the face $F_i$ is $i$-dimensional (see for example [18] for more information on regular polytopes). We can think of the flags from Figure 1.3 as partial flags compared to the 'complete' flags.

The symmetry group of a regular polytope has a very strong notion of transitivity to it. We will see that relaxing this strong notion and considering other types of transitivity gives an interpretation of designs in the groups $C_r \wr S_n$ (which are the symmetry groups of regular polytopes). From there, one can give a more algebraic interpretation of designs and extend these ideas to the groups $G \wr S_n$ where $G$ is an arbitrary finite abelian group.

## ORIGINAL CONTRIBUTIONS - PART 1

For each of the notions of transitivity that we consider in the group $G \wr S_n$, we show that they can be characterised as designs in the conjugacy class scheme of $G \wr S_n$. Then, we use our characterisation to compare different types of transitivity. Our results are generalisations of the results by Martin and Sagan from [46].

For this introduction, we only consider the case $G = C_r$ and consider regular polytopes, for example the hypercube for $G = C_2$. Let $k$ be an integer with $0 \leqslant k \leqslant n$ and $\sigma = (\sigma_1, \sigma_2, \ldots)$ be a sequence of positive integers that sum to $n - k$ (so $\sigma$ is a *composition* of $n - k$, not necessarily a partition of $n - k$). We call a chain of faces $F_0 \subseteq F_1 \subseteq \ldots$ of the polytope a $(\sigma, k)$-*flag* if $\dim F_0 = k$ and $\dim F_i - \dim F_{i-1} = \sigma_i$ for every $i$. Using this notation, the examples from Figure 1.3 are $(21, 0)$-, $(11, 1)$- and $(12, 0)$-flags, respectively. The complete flags are $(111, 0)$-flags.

We call a subgroup $Y$ of the symmetry group of a regular polytope $(\sigma, k)$-*transitive* if it acts transitively on the set of all $(\sigma, k)$-flags. We shall later see that the order of the parts of $\sigma$ does not matter. That is, a subgroup $Y$ is $(\sigma, k)$-transitive if and only if it is $(\tilde{\sigma}, k)$-transitive where $\tilde{\sigma}$ is a permutation of $\sigma$. Different permutations $\tilde{\sigma}$ of $\sigma$ result in different geometric interpretations of $(\sigma, k)$-transitivity, but all of them are equivalent.

Our first main result is that there is a relation denoted by $\to$ that is given by an algorithm that takes a transitivity type $(\sigma, k)$ as input and outputs a set $T$ such that $(\sigma, k)$-transitive sets are precisely $T$-designs. It is a special case of Theorem 5.3.11.

**Theorem 1.1.2.** *A subgroup $Y$ of $C_r \wr S_n$ is $(\sigma, k)$-transitive if and only if it is a $T$-design where the set $T$ is given by $(\sigma, k) \to T$.*

This gives a characterisation of $(\sigma, k)$-transitive subgroups as designs in the conjugacy class scheme of $C_r \wr S_n$. We will explain this theorem in more detail in Chapter 5 where we prove a more general version that applies to subsets instead of only subgroups, works for the wreath products $G \wr S_n$ where $G$ is a finite abelian group and applies to more types of transitivity than just transitivity on $(\sigma, k)$-flags. It will be helpful to keep the transitivity types from Figure 1.3 in mind as simple examples of the notions of transitivity that we deal with later.

Another main result is that we have a full classification of $(\sigma, k)$-transitivity. That is, given a $(\sigma, k)$-transitive subgroup $Y$, we know exactly for which types $(\tau, \ell)$ the group $Y$ is also $(\tau, \ell)$-transitive. The following theorem is a special case of Theorem 5.5.11. In the theorem, $\sigma \cup (k)$ denotes the partition obtained by adding a part of size $k$ to $\sigma$, and $\trianglelefteq$ denotes the dominance order (explained in Section 2.2). In the case $k = 0$, we have $\sigma \cup (k) = \sigma$.

**Theorem 1.1.3.** *Let $Y$ be a $(\sigma, k)$-transitive subgroup of $C_r \wr S_n$. Then $Y$ is also $(\tau, \ell)$-transitive if $k \leqslant \ell$ and $(\sigma \cup (k)) \trianglelefteq (\tau \cup (\ell))$.*

As a sample application of this theorem, we obtain the (perhaps surprising) statement that a group $Y$ that acts transitively on the edges of a 3-dimensional cube also acts transitively on its faces (see Example 5.5.12). Again, we will see in Chapter 5 that analogous results also hold for subsets and groups of the form $G \wr S_n$ where $G$ is finite and abelian. It will be helpful to keep the simple example of edge- and face-transitivity in mind when dealing with more general results later.

In Section 5.6 and Section 5.7, we investigate $(\sigma, n-t)$-transitivity in the group $C_r \wr S_n$ in more detail where $\sigma = 1^t$ is the partition of $t$ containing $t$ parts of size 1. One can think of $(1^t, n-t)$-transitivity as the analogue of $t$-transitivity in the symmetric group. We obtain the following application of our results to finite projective planes.

**Theorem 1.1.4.** *Let $q$ be a prime power. If there is no sharply $(1^t, n-t)$-transitive set in $C_{q-1} \wr S_q$, then there is no finite projective plane of order $q-1$. If there is no sharply $(1^t, n-t)$-transitive set in $C_q \wr S_{q+1}$, then there is no finite projective plane of order $q+1$.*

Our results on the generalised symmetric group are publicly available as a preprint.

[35] L. Klawuhn and K.-U. Schmidt. *Transitivity in wreath products with symmetric groups*. Preprint, arXiv. 2024. URL: https://arxiv.org/abs/2409.20495

Both authors are first authors with equal rights and equally contributed to the research on the project.

### DESIGNS OF PERFECT MATCHINGS

In addition to the conjugacy class scheme of the generalised symmetric group, we also study the perfect matching association scheme. Here, the hyperoctahedral group $C_2 \wr S_n$ appears again, but this time as a subgroup of the symmetric group $S_{2n}$.

A *perfect matching* of the complete graph $K_{2n}$ is a set of $n$ independent edges. In graph theory, it is also called a 1-*factor*. It is well-known that the complete graph $K_{2n}$ can be decomposed into $2n-1$ perfect matchings for every $n \in \mathbb{N}$ (see for example [30, Theorem 9.1]). Such a decomposition is called a 1-*factorisation*. These structures satisfy properties very similar to block designs and are examples of designs in the perfect matching association scheme.

Perfect matchings have been investigated over 100 years ago already and 1-factorisations can be used to schedule round robin tournaments such as chess tournaments or a football league. Constructions appeared in the 19th century, for example in the book *Récréations mathématiques* by Édouard Lucas [43] (phrased in terms of children dancing in a circle) or in an article in Deutsche Schachzeitung [56], proposing a way to schedule chess games according to a mathematical rule. However, the connection to tournaments makes it very likely that 1-factorisations have been studied much earlier, even without written record.

In a 1-factorisation, every edge appears in exactly one perfect matching. In light of our general idea that a design in an association scheme 'approximates' the whole set, we can see how this can be interpreted in the association scheme of perfect matchings. Taking *all* perfect matchings of $K_{2n}$, it is clear that every edge appears in a constant number of them. A 1-factorisation has the same property with the constant being 1. Phrased differently, a 1-factorisation contains each edge the same number of times. Thus, we can think of a 1-factorisation as the analogue of a block design of strength 1 where every point appears in precisely 1 block (even though a 1-factorisation is not a block design in the actual sense). Figure 1.4 gives an example of a 1-factorisation of $K_6$.

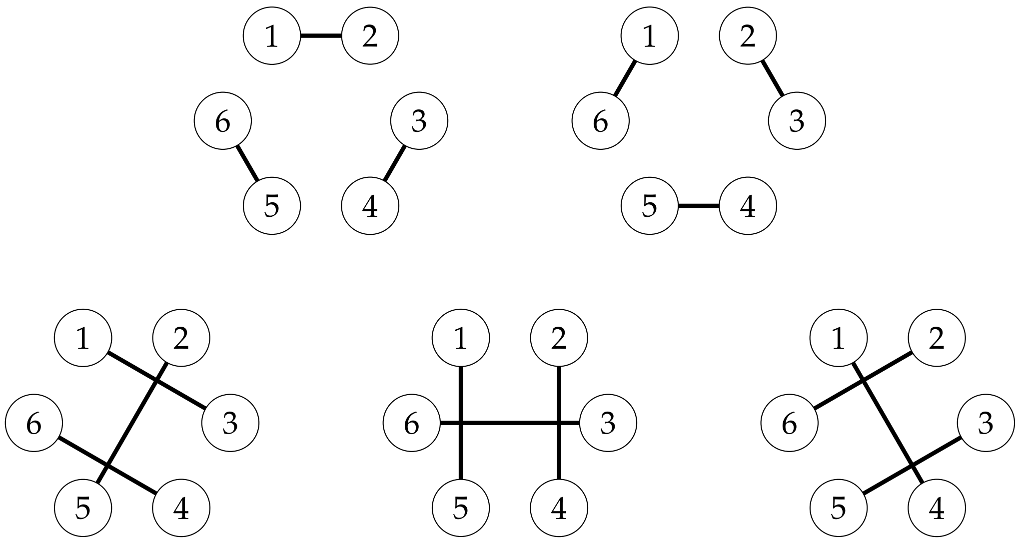



Figure 1.4: A 1-factorisation of $K_6$

We can generalise this idea to more than 1 edge. Just like the blocks of a block design of strength $t$ contain each set of $t$ distinct elements constantly often, we can think of a design in the perfect matching association scheme as a set of perfect matchings containing each set of $t$ distinct edges constantly often. Clearly, we have to restrict ourselves to sets of $t$ distinct *independent* edges because perfect matchings do not contain dependent edges. Taking $t = 2$, we obtain the definition of a *hyperfactorisation*, studied by Boros, Jungnickel and Vanstone [8]. They showed that for $n \geqslant 5$, the complete graph $K_{2n}$ always has a non-trivial hyperfactorisation where non-trivial means that the hyperfactorisation is not the set of *all* perfect matchings of $K_{2n}$.

Cameron [12] generalised hyperfactorisations to partition systems. He defines an $s$-$(t,n)$ *partition system* to be a collection $S$ of partitions of a set $X$ of size $n$ into subsets of size $t$ having the property that given any $s$ disjoint subsets of size $t$ of $X$, there is a unique partition in $S$ which has all of the given subsets as parts. This is a natural way to generalise 1-factorisations and hyperfactorisations of complete graphs. Indeed, a 1-factorisation of $K_n$ is simply a 1-$(2,n)$ partition system. Independently, Stinson [60] defined *hyperresolutions* of a resolvable block design, which turn out to be 2-$(t,n)$ partition systems. Cameron constructed infinitely many examples of 2-$(2,n)$ partition systems from hyperovals of projective planes of even order.

In this thesis, we investigate generalisations of these concepts and show that structures with these properties are designs in the perfect matching association scheme.

## ORIGINAL CONTRIBUTIONS - PART 2

We generalise the notion of $s$-$(2,n)$ partition systems to $\lambda$-*factorisations* by replacing the parameter $s$ with a partition $\lambda$ and use the theory of association schemes to investigate them. In an $s$-$(2,n)$ partition system, any set of $s-1$ subsets of size 2 appears a constant number of times. The counting involved does not give rise to restrictions on the parameters $s$ and $n$ (see [12, Theorem 7.2]). We show that similar counting arguments work for $\lambda$-factorisations and give rise to non-trivial divisibility conditions by comparing different partitions $\lambda$.

Work on the association scheme of perfect matchings of the complete graph $K_{2n}$ has been done by Rands [52]. He calculated the eigenvalues of the association scheme for $n \in \{4,5,6\}$, and he also observed that we can view abstract hyperovals and 1-factorisations as cliques for this association scheme. We take this viewpoint further and consider $\lambda$-factorisations as designs in the association scheme, and use Delsarte theory to produce results on the existence of such designs.

Our main result is a characterisation of $\lambda$-factorisations as $T$-designs (see Theorem 6.2.4).

**Theorem 1.1.5.** *A non-empty set of perfect matchings is a $\lambda$-factorisation if and only if it is a $T$-design where $T$ is the set of all partitions $\mu$ of $n$ with $\lambda \trianglelefteq \mu \neq (n)$.*

Then, we use this characterisation to prove several interesting corollaries of this result. One of them is the following comparison of different $\lambda$-factorisations (see Theorem 6.3.1).

**Theorem 1.1.6.** *Let $D$ be a $\lambda$-factorisation. If $\mu \trianglerighteq \lambda$, then $D$ is also a $\mu$-factorisation.*

As a corollary, we obtain divisibility conditions on the parameter $n$ such that a $\lambda$-factorisation of $K_{2n}$ can exist.

Furthermore, in Corollary 6.3.12 we show that there is no $(2, 1, \ldots, 1)$-factorisation for all $n \geqslant 4$, generalising a result of Cameron [12, Theorem 7.6] on partition systems.

The article about our results on the association scheme of perfect matchings was published in the journal *Algebraic Combinatorics* and is publicly available (open access).

[3] J. Bamberg and L. Klawuhn. "On the association scheme of perfect matchings and their designs". In: *Algebraic Combinatorics* 9.3 (2026), pp. 789–809. DOI: 10.5802/alco.490

Both authors are first authors with equal rights. The first author contributed the computations for small $n$ and the GAP code in the appendix (see a.2). The second author contributed the representation theoretic calculations and the initial idea for the project. Both authors contributed equally to the remaining parts of the project.

## OUTLINE OF THIS THESIS

Here is how we structure this thesis. We start by giving an overview of representation and character theory of finite groups in Chapter 2, especially the representation theory of the symmetric group. This is one of two important areas where our techniques come from. The other is the theory of association schemes which we present in Chapter 3. We give a detailed introduction to the theory with special emphasis on codes and designs in an association scheme. Special attention is paid to conjugacy class schemes where the character theoretic results from Chapter 2 come into play.

Having completed the foundation, we describe the well-known conjugacy class scheme of the symmetric group and its designs in Chapter 4. These results can be thought of as a template for the kind of results obtained in later chapters.

In Chapter 5, we generalise results from Chapter 4 by answering questions from [46] in a vastly generalised form. Constructions of designs in the generalised symmetric group and connections to orthogonal polynomials and finite projective planes are given.

In Chapter 6, we apply similar ideas to the perfect matching association scheme. We obtain a characterisation of designs and show that some objects studied in finite geometry are in fact special cases of designs in the perfect matching association scheme that we study. At the end of Chapters 5 and 6, we list some open problems.

All computations were done using GAP [26] which has been a very useful tool during research.

# 2

# REPRESENTATION AND CHARACTER THEORY

We start our journey with an introduction to representation and character theory. Representation theory can be a powerful tool to deduce combinatorial results in groups, especially results associated with group actions. Methods from representation theory underpin many of our results, both in the generalised symmetric group and in the perfect matching association scheme.

In this chapter, we give an introduction to the theory. We follow [57], where we also refer the reader for a detailed treatment of the subject and proofs of the statements in this chapter. In Section 2.1, we give an overview of the definitions and results from representation theory that will be needed in this thesis. In Section 2.2, we deal with the representation theory of the symmetric group $S_n$, especially its irreducible characters.

## 2.1 DEFINITIONS, EXAMPLES AND BASIC RESULTS

Consider the following matrices over an arbitrary field $\mathbb{F}$:

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$
$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad D = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad E = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is easy to see that the set $\{I, A, B, C, D, E\}$ forms a group together with matrix multiplication that is isomorphic to $S_3$. Hence, we have represented the permutation group $S_3$ using linear maps of the 3-dimensional vector space $\mathbb{F}^3$. This is the key idea of representation theory. It allows us to use techniques of linear algebra to investigate groups.

In what follows, $G$ will always be a finite group and $V$ and $W$ are non-zero finite dimensional vector spaces over $\mathbb{C}$. The *complex general linear group* of $V$ is denoted by $\mathrm{GL}(V)$ and the complex general linear group of $\mathbb{C}^n$ is denoted by $\mathrm{GL}(n, \mathbb{C})$. It is the group of all invertible $n \times n$ matrices with complex entries.

**Definition 2.1.1.** A *representation* of $G$ in $V$ is a group homomorphism

$$\rho : G \to \mathrm{GL}(V), g \mapsto \rho(g).$$

Sometimes, we write $\rho_g$ for $\rho(g)$. We call $\dim(V)$ the *degree* of the representation and denote it by $\deg(\rho)$.

In a representation, every group element $g \in G$ is associated with an isomorphism $\rho_g$ of a vector space $V$.

**Example 2.1.2.** Let $G = S_3$ be the symmetric group on 3 elements. Recalling the matrices from earlier, the map $\rho : S_3 \to \mathrm{GL}(3, \mathbb{C})$ defined by

$$\begin{aligned} \rho\big(\mathrm{id}\big) &= I, & \rho\big((1,2,3)\big) &= A, & \rho\big((1,3,2)\big) &= B, \\ \rho\big((2,3)\big) &= C, & \rho\big((1,3)\big) &= D, & \rho\big((1,2)\big) &= E \end{aligned}$$

is a group homomorphism. Hence, it defines a representation of $S_3$ in $\mathbb{C}^3$.

**Example 2.1.3.** The map $\rho : G \to \mathrm{GL}(1, \mathbb{C})$ sending every group element to the $1 \times 1$ matrix $(1)$ is a representation of $G$, called the *trivial* representation. It has degree 1.

Although innocent-looking, the trivial representation plays an important role when dealing with *permutation representations*. Example 2.1.2 is such a permutation representation. We now define it for an arbitrary group action. If $G$ acts on a set $\Omega$, then for $g \in G$ and $x \in \Omega$, the group action is denoted by $g \cdot x$.

**Example 2.1.4.** Let $G$ be a finite group acting on a finite set $\Omega$ and denote by $V$ the vector space generated by a basis $(e_x)_{x \in \Omega}$ that is indexed by the elements of $\Omega$. For an element $g \in G$, let $\rho_g : V \to V$ be the linear map that sends $e_x$ to $e_{g \cdot x}$ for every $x \in \Omega$. Then the map $\Xi : G \to \mathrm{GL}(V)$ with $\Xi(g) = \rho_g$ is a representation of $G$, called the *permutation representation*. Its degree is $\deg(\Xi) = |\Omega|$.

We shall later see how permutation representations of transitive group actions can be obtained from trivial representations of subgroups of $G$.

It is well-known that every linear map between finite dimensional vector spaces can be described using a matrix by picking a basis. Doing so, we obtain a representation in $\mathrm{GL}(n, \mathbb{C})$ which is essentially the same as the original one. This suggests the following definition.

**Definition 2.1.5.** Let $\rho : G \to \mathrm{GL}(V)$ and $\widetilde{\rho} : G \to \mathrm{GL}(W)$ be two representations of $G$. If there exists an isomorphism $T : V \to W$ such that

$$\widetilde{\rho}(g) = T \circ \rho(g) \circ T^{-1}$$

for every $g \in G$, then the representations $\rho$ and $\widetilde{\rho}$ are called *equivalent*.

We find that a representation $\rho : G \to \mathrm{GL}(V)$ is equivalent to a representation $\tilde{\rho} : G \to \mathrm{GL}(\dim(V), \mathbb{C})$. Hence, we can think of representations using linear maps and representations using matrices as being essentially the same. This allows us to use the different notions interchangeably, depending on which is more convenient in a given situation.

We are interested in decomposing a given representation into smaller *subrepresentations* that are *irreducible*. Let us define these notions.

**Definition 2.1.6.** Let $\rho : G \to \mathrm{GL}(V)$ be a representation and let $W$ be a subspace of $V$. The subspace $W$ is called *invariant under* $G$ or $G$*-invariant* if $\rho_g(W) \subseteq W$ for every $g \in G$.

It is immediate that the restriction of an isomorphism $\rho_g$ to a $G$-invariant subspace $W$ yields an isomorphism of $W$ for every $g \in G$, hence an element of $\mathrm{GL}(W)$. Thus, we obtain a representation of $G$ in $W$. This motivates the following definition.

**Definition 2.1.7.** Let $\rho : G \to \mathrm{GL}(V)$ be a representation and let $W$ be a $G$-invariant subspace of $V$. Then the restriction

$$\rho_{|W} : G \to \mathrm{GL}(W), \quad g \mapsto (\rho_g)_{|W}$$

is called a *subrepresentation* of $\rho$.

**Example 2.1.8.** Consider the representation of $S_3$ from Example 2.1.2. The matrices $\rho(\sigma)$ clearly fix the vector $(1,1,1)^T$ for every $\sigma \in S_3$. Hence, the vector space $W = \langle (1,1,1)^T \rangle$ of dimension 1 is invariant under the action of $S_3$ and thus gives rise to a subrepresentation of $\rho$.

The following theorem is crucial (see [57, Ch. 2 Th. 1] for a proof).

**Theorem 2.1.9.** *Let $\rho : G \to \mathrm{GL}(V)$ be a representation of $G$ in $V$ and let $W$ be a $G$-invariant subspace of $V$. Then there exists a complement $\widetilde{W}$ of $W$ in $V$ that is invariant under $G$.*

This suggests that we can break down a representation into smaller representations, writing it as a sum of subrepresentations. We make this more explicit with the following definition.

**Definition 2.1.10.** Let $\rho : G \to \mathrm{GL}(V)$ and $\tilde{\rho} : G \to \mathrm{GL}(W)$ be two representations. The *direct sum* of $\rho$ and $\tilde{\rho}$ is the representation

$$\rho \oplus \tilde{\rho} : G \to \mathrm{GL}(V \oplus W), \quad (\rho \oplus \tilde{\rho})_g(v, w) = \big(\rho_g(v), \tilde{\rho}_g(w)\big)$$

for every $g \in G$ and every $(v, w) \in V \oplus W$.

Theorem 2.1.9 says that if $\rho$ is a representation of $G$ in $V$ and we find a $G$-invariant subspace of $V$, then we can write $\rho$ as the direct sum of two subrepresentations. This leads to the following important definition.

**Definition 2.1.11.** A representation $\rho : G \to GL(V)$ is called *irreducible* if $V$ is non-zero and the only $G$-invariant subspaces of $V$ are the trivial subspaces $\{0\}$ and $V$.

In light of Theorem 2.1.9, a representation $\rho$ of $G$ in $V$ is irreducible if and only if it cannot be written as the direct sum of two non-zero subrepresentations of $\rho$.

**Example 2.1.12.** The representation $\rho$ from Example 2.1.2 is not irreducible because the subspace $W = \langle (1,1,1)^T \rangle$ is $S_3$-invariant. It is easy to see that the orthogonal complement $W^\perp$ under the standard inner product of $\mathbb{C}^3$ is also $S_3$-invariant and it turns out that it is irreducible. Hence,

$$\rho = \rho_{|W} \oplus \rho_{|W^\perp}$$

is the decomposition of $\rho$ into irreducible subrepresentations.

Furthermore, we observe that every representation of degree 1 is irreducible.

We have the following important theorem, often called *Maschke's Theorem*.

**Theorem 2.1.13.** *Every representation $\rho$ can be decomposed into a direct sum of irreducible representations.*

Having laid the groundwork, we now turn to a different way to view representations and their decompositions. Every representation comes with a so called *character* that carries valuable information about a representation.

**Definition 2.1.14.** Let $\rho : G \to \mathrm{GL}(V)$ be a representation of $G$. Then the function

$$\chi : G \to \mathbb{C}, \quad \chi(g) = \mathrm{Tr}(\rho(g))$$

is called the *character* of $\rho$. Here, Tr denotes the trace function. Furthermore, if $\rho$ is irreducible, then we call $\chi$ an *irreducible character*.

When interpreting representations in terms of matrices, the trace of a matrix is simply the sum of its diagonal entries. Notice that $\rho(e) = I_n$, the $n \times n$ identity matrix, so for the corresponding character $\chi$ we have $\chi(e) = \deg(\rho)$.

**Example 2.1.15.** The character of the trivial representation is called the *trivial character*, denoted by $1_G$. It is irreducible and satisfies $1_G(g) = 1$ for every $g \in G$.

Furthermore, every representation of degree 1 is essentially equal to its character (via interpreting $1 \times 1$ matrices as complex numbers). If $\chi$ is the character of a degree 1 representation, then it is irreducible and satisfies $\chi(gh) = \chi(g)\chi(h)$ for every $g, h \in G$. Characters with that property are called *linear characters*.

Characters are important because they uniquely determine a representation as the following theorem shows (see for example [57, Ch. 2 Cor. 2]).

**Theorem 2.1.16.** *Let $\rho : G \to \mathrm{GL}(V)$ and $\widetilde{\rho} : G \to \mathrm{GL}(W)$ be two representations and let $\chi$ and $\widetilde{\chi}$ be their characters. Then $\rho$ and $\widetilde{\rho}$ are equivalent if and only if $\chi = \widetilde{\chi}$.*

We can thus talk about representations and characters interchangeably. Their connection also manifests in how representations and characters can be decomposed into irreducibles.

**Theorem 2.1.17.** *Let $\rho : G \to \mathrm{GL}(V)$ and $\widetilde{\rho} : G \to \mathrm{GL}(W)$ be two representations and let $\chi$ and $\widetilde{\chi}$ be their characters. Then the character of $\rho \oplus \widetilde{\rho}$ is equal to $\chi + \widetilde{\chi}$.*

Hence, characters can be decomposed into irreducible characters just like representations can be decomposed into irreducible representations.

**Example 2.1.18.** Consider the representation of $S_3$ from Example 2.1.2 and let $\chi$ be its character. Taking the traces, we compute

$$\begin{aligned} \chi(\mathrm{id}) &= 3, & \chi\big((1,2,3)\big) &= 0, & \chi\big((1,3,2)\big) &= 0,\\ \chi\big((2,3)\big) &= 1, & \chi\big((1,3)\big) &= 1, & \chi\big((1,2)\big) &= 1. \end{aligned}$$

Furthermore, the restriction of $\rho$ to the subspace $W = \langle (1,1,1)^T \rangle$ is the trivial representation. We denote its character by $\chi_W$ and it satisfies $\chi_W(\sigma) = 1$ for all $\sigma \in S_3$. From Theorem 2.1.16 and Theorem 2.1.17, it follows that $\chi = \chi_W + \chi_{W^\perp}$ where $\chi_{W^\perp}$ is the character of the restriction of $\rho$ to $W^\perp$. Thus, we have $\chi_{W^\perp} = \chi - \chi_W$ and we find

$$\begin{aligned} \chi_{W^\perp}(\mathrm{id}) &= 2, & \chi_{W^\perp}\big((1,2,3)\big) &= -1, & \chi_{W^\perp}\big((1,3,2)\big) &= -1,\\ \chi_{W^\perp}\big((2,3)\big) &= 0, & \chi_{W^\perp}\big((1,3)\big) &= 0, & \chi_{W^\perp}\big((1,2)\big) &= 0. \end{aligned}$$

It is easy to see that $W^\perp$ has no non-trivial $S_3$-invariant subspaces. Hence, the restriction of $\rho$ to $W^\perp$ is irreducible and $\chi_{W^\perp}$ must be an irreducible character of $S_3$. Notice that $\chi_{W^\perp}(\mathrm{id}) = 2$, agreeing with $W^\perp$ being 2-dimensional.

The character $\chi$ is an example of a *permutation character*. The decomposition of permutation characters will be of utmost importance in this thesis.

**Definition 2.1.19.** Let $G$ be a finite group acting on a set $\Omega$. The function $\xi : G \to \mathbb{C}$ defined by

$$\xi(g) = |\mathrm{Fix}(g)| = |\{x \in \Omega \mid g \cdot x = x\}|$$

is called the *permutation character* of $G$ on $\Omega$.

Example 2.1.18 shows many patterns that are true for characters in general, for example constant values on conjugacy classes. We develop the theory now (following [57, Ch. 2]).

**Definition 2.1.20.** Let $G$ be a group and $g \in G$. Then the set

$$C_g = \{hgh^{-1} \in G \mid h \in G\}$$

is called the *conjugacy class* of $g$. Elements that lie in the same conjugacy class are called *conjugate*.

It is well-known that conjugacy defines an equivalence relation, hence the conjugacy classes partition the group. Conjugate elements of a group often have similar properties.

**Theorem 2.1.21.** *Let $G$ be a finite group and $\chi$ be a character of $G$. Then $\chi$ is constant on the conjugacy classes of $G$. In other words, for every $g, h \in G$ we have*

$$\chi(g) = \chi(hgh^{-1}).$$

See [57, Ch. 2 Prop. 1] for a proof. Functions from $G$ to $\mathbb{C}$ that are constant on conjugacy classes are called *class functions*. By Theorem 2.1.21, characters are class functions, hence there is no ambiguity if we write $\chi(C)$ for the value of a character $\chi$ on a conjugacy class $C$.

**Example 2.1.22.** The conjugacy classes of $S_3$ are

$$\{\mathrm{id}\},\ \{(1,2,3),(1,3,2)\} \text{ and } \{(1,2),(1,3),(2,3)\}.$$

We see that the characters in Example 2.1.18 are constant on the conjugacy classes.

We denote by $\mathbb{C}\Omega$ the space of all functions $f : \Omega \to \mathbb{C}$. It has an inner product defined by

$$\langle f, g\rangle = \frac{1}{|G|}\sum_{x\in G} f(x)\overline{g(x)}$$

for every $f, g \in \mathbb{C}\Omega$. Characters are elements of $\mathbb{C}G$. It turns out that irreducible characters have special properties with regard to this inner product.

**Theorem 2.1.23.** *Let $G$ be a finite group and $\chi$ and $\widetilde{\chi}$ be two distinct irreducible characters of $G$. Then we have*

$$\langle \chi, \widetilde{\chi} \rangle = \begin{cases} 1, & \chi = \widetilde{\chi}, \\ 0, & \textit{otherwise}. \end{cases}$$

*That is, the irreducible characters from an orthonormal system with respect to the inner product $\langle \cdot, \cdot \rangle$. In fact, the irreducible characters form an orthonormal basis of the space of all class functions on $G$.*

See [57, Ch. 2 Th. 3] for a proof. Notice that the fact that the irreducible characters are a basis of the space of class functions implies that their number is equal to the number of conjugacy classes of $G$. In particular, a finite group has finitely many irreducible characters.

We now come to one of the most important theorems about the decomposition of characters into irreducibles (see [57, Ch. 2 Th. 4]).

**Theorem 2.1.24.** *Let $G$ be a finite group and $\rho$ be a representation of $G$ in $V$ with character $\chi$. Suppose that $V$ decomposes into a direct sum of irreducible representations*

$$V = W_1 \oplus \ldots \oplus W_k.$$

*Then, for any irreducible representation $W$ with character $\chi_W$, the number of $W_i$ isomorphic to $W$ is equal to the scalar product $\langle \chi, \chi_W \rangle$.*

This theorem has the important consequence that the decomposition of a representation (resp. a character) into irreducibles is unique up to equivalence. If $\chi_1, \ldots, \chi_k$ are the irreducible characters of $G$ and $\chi$ is any character of $G$, then we can write

$$\chi = \langle \chi, \chi_1 \rangle \chi_1 + \ldots + \langle \chi, \chi_k \rangle \chi_k.$$

The numbers $m_i = \langle \chi, \chi_i \rangle$ are non-negative integers and are equal to the number of times the irreducible character $\chi_i$ is contained in $\chi$. They are called *multiplicities*. The irreducible characters $\chi_i$ with $m_i > 0$ are called *irreducible constituents* of $\chi$. In general, we have

$$\langle \chi, \chi \rangle = m_1^2 + \ldots + m_k^2.$$

Notice that Theorem 2.1.24 also gives a criterion for a character to be irreducible.

**Theorem 2.1.25.** *Let $G$ be a finite group and $\chi$ be a character of a representation $\rho$ of $G$. Then $\rho$ is irreducible if and only if $\langle \chi, \chi \rangle = 1$.*

**Example 2.1.26.** Consider the permutation character of $S_3$ from Example 2.1.18 and its decomposition into the irreducible characters $\chi_W$ and $\chi_{W^\perp}$. First, we compute

$$\langle \chi, \chi \rangle = \frac{1}{6}\left(3^2 + 0^2 + 0^2 + 1^2 + 1^2 + 1^2\right) = 2,$$

agreeing with the fact that $\chi$ is not irreducible. On the other hand, we find

$$\langle \chi_W, \chi_W \rangle = \frac{1}{6}\left(1^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2\right) = 1,$$
$$\langle \chi_{W^\perp}, \chi_{W^\perp} \rangle = \frac{1}{6}\left(2^2 + (-1)^2 + (-1)^2 + 0^2 + 0^2 + 0^2\right) = 1,$$

proving again that $\chi_W$ and $\chi_{W^\perp}$ are irreducible characters of $S_3$.

Since $S_3$ has 3 conjugacy classes, there is one more irreducible character of $S_3$. It is easily seen that the sign $\mathrm{sg}(\sigma)$ of a permutation $\sigma \in S_3$ is constant on conjugacy classes. Since sg is a homomorphism from $S_3$ to $\mathbb{C}$, it is a 1-dimensional representation and hence irreducible. It follows that the sign character $\chi_{\mathrm{sg}}$ with $\chi_{\mathrm{sg}}(\sigma) = \mathrm{sg}(\sigma)$ for every $\sigma \in S_3$ is an irreducible character of $S_3$. Hence, we have found all irreducible characters of $S_3$.

Here is an overview of the character values of the symmetric group $S_3$. For the columns, we give one representative per conjugacy class. We denote by $1_{S_3}$ the trivial character (which is $\chi_W$ above) and by $\chi_2$ the 2-dimensional irreducible character $\chi_{W^\perp}$ from above.

| | id | $(1,2)$ | $(1,2,3)$ |
|---|---|---|---|
| $1_{S_3}$ | 1 | 1 | 1 |
| $\chi_{\mathrm{sg}}$ | 1 | $-1$ | 1 |
| $\chi_2$ | 2 | 0 | $-1$ |

The irreducible characters of the symmetric group are studied in more detail in Section 2.2.

Next, we study two constructions for characters called restriction and induction. Recall that a representation $\rho$ of a group $G$ in a vector space $V$ can be restricted to a subspace of $V$. Similarly, we can restrict $\rho$ to a subgroup $H$ of $G$. Since our focus in later chapters will be on characters, we define these notions for characters instead of representations.

**Definition 2.1.27.** Let $G$ be a finite group, let $\chi$ be a character of $G$ and let $H$ be a subgroup of $G$. The *restriction* of $\chi$ to $H$ is denoted by $\chi\downarrow_H^G$ and given by

$$\chi\downarrow_H^G\colon H \to \mathbb{C}, \quad h \mapsto \chi(h).$$

We call $\chi\downarrow_H^G$ a *restricted character*.

It is easy to see that $\chi \downarrow_H^G$ is indeed a character of $H$ and it has the same degree as $\chi$. However, it is important to note that an irreducible character of $G$ need not stay irreducible after restriction to a subgroup.

Given a character of a finite group $G$, restriction produces a character of a subgroup $H$ of $G$. There is an operation doing the opposite, producing a character of $G$ given a character of a subgroup $H$ of $G$.

**Definition 2.1.28.** Let $G$ be a finite group, let $H$ be a subgroup of $G$ and let $\chi$ be a character of $H$. The *induction* of $\chi$ to $G$ is denoted by $\chi \uparrow_H^G$ and given by

$$\chi \uparrow_H^G : G \to \mathbb{C}, \quad g \mapsto \frac{1}{|H|} \sum_{\substack{x \in G: \\ xgx^{-1} \in H}} \chi(x^{-1}gx).$$

We call $\chi \uparrow_H^G$ an *induced character*. Many properties of induced characters can be found in [57, Ch. 7.1].

The induced character $\chi \uparrow_H^G$ is indeed a character of $G$. Its degree is $\frac{|G|}{|H|} \deg(\chi)$. However, similar to restriction, we point out that an irreducible character of $H$ need not stay irreducible after induction to $G$.

**Example 2.1.29.** Consider for one more time the representation of $S_3$ from Example 2.1.18 and its character $\chi$. Let $H = \mathrm{Stab}(1) = S_{\{2,3\}} = \{\mathrm{id}, (2,3)\}$. We show that $\chi = 1 \uparrow_H^{S_3}$.

The induced character $1 \uparrow_H^{S_3}$ is given by the formula

$$1 \uparrow_H^{S_3} (\sigma) = \frac{1}{2} \sum_{\substack{\tau \in S_3: \\ \tau\sigma\tau^{-1} \in H}} 1.$$

Remember that characters are constant on conjugacy classes, so we only have to evaluate $1 \uparrow_H^{S_3}$ on the elements $\mathrm{id}, (1,2)$ and $(1,2,3)$. We find

$$1 \uparrow_H^{S_3} (\mathrm{id}) = \frac{1}{2} \sum_{\substack{\tau \in S_3: \\ \tau \,\mathrm{id}\, \tau^{-1} \in H}} 1 = \frac{1}{2} \sum_{\tau \in S_3} 1 = \frac{|S_3|}{2} = 3.$$

Similarly, we find the values $1 \uparrow_H^{S_3} ((1,2)) = 1$ and $1 \uparrow_H^{S_3} ((1,2,3)) = 0$. This proves $1 \uparrow_H^{S_3} = \chi$.

This example can be generalised to any transitive group action. It turns out that every permutation character of such an action is in fact obtained as the induction of a trivial character (see [57, Ex. 7.2]).

**Theorem 2.1.30.** *Let $G$ be a finite group acting transitively on a set $\Omega$ and let $\xi_\Omega$ be the permutation character of $G$ on $\Omega$. Consider an element $x \in \Omega$ and let $H = \mathrm{Stab}(x)$ be its stabiliser. Then we have*

$$\xi_\Omega = 1 \uparrow_H^G .$$

*In particular, the character $1 \uparrow_H^G$ is the permutation character of the action of $G$ on the left cosets $gH$ of $G$ by left multiplication.*

This fact is crucial when decomposing permutation characters. We call the induced character $1 \uparrow_H^G$ the *permutation character of the subgroup $H$*. It counts the number of fixed points of the action of $G$ on the cosets of $H$.

Restriction and induction can be thought of as duals of each other because they behave like adjoints with respect to the inner product $\langle \cdot, \cdot \rangle$. This fact is also called *Frobenius reciprocity*.

**Theorem 2.1.31** (Frobenius Reciprocity)**.** *Let $G$ be a finite group and let $H$ be a subgroup of $G$. Furthermore, let $\chi$ be a character of $H$ and let $\psi$ be a character of $G$. Then we have*

$$\langle \chi \uparrow_H^G, \psi \rangle_G = \langle \chi, \psi \downarrow_H^G \rangle_H$$

*where $\langle \cdot, \cdot \rangle_G$ and $\langle \cdot, \cdot \rangle_H$ are the inner products on the vector spaces $\mathbb{C}G$ and $\mathbb{C}H$, respectively.*

This theorem is extremely helpful when decomposing characters.

**Example 2.1.32.** Let $G$ be a finite group and consider the action of $G$ on itself by left multiplication. The permutation representation of this action is called the *(left) regular representation*. Its character is the permutation character $\xi$ of this action, that is the number of fixed points, and is given by

$$\chi(e) = |G|, \quad \chi(g) = 0 \text{ for } g \neq e.$$

We see that the stabiliser of any element is trivial and contains only the identity element $e \in G$. Hence, from Theorem 2.1.30, we find that $\xi = 1 \uparrow_{\{e\}}^G$.

Now let $\chi$ be an irreducible character. Using Theorem 2.1.31, we calculate

$$\langle \xi, \chi \rangle_G = \langle 1 \uparrow_{\{e\}}^G, \chi \rangle_G = \langle 1_{\{e\}}, \chi \downarrow_{\{e\}}^G \rangle_{\{e\}} = \frac{1}{|\{e\}|} 1_{\{e\}}(e)\overline{\chi(e)} = \deg(\chi).$$

It follows that the regular representation contains every irreducible character $\chi$ with multiplicity $\deg(\chi)$. Denoting by $\mathrm{Irr}(G)$ the set of all irreducible characters of $G$, we can write

$$\xi = \sum_{\chi \in \mathrm{Irr}(G)} \deg(\chi)\, \chi.$$

The degree of the regular representation is $\deg(\xi) = |G|$. As an immediate consequence, we obtain the formula

$$|G| = \xi(e) = \sum_{\chi \in \mathrm{Irr}(G)} \deg(\chi)\, \chi(e) = \sum_{\chi \in \mathrm{Irr}(G)} \deg(\chi)^2.$$

Restriction and induction are transitive operations in the following sense.

**Theorem 2.1.33.** *Let $G$, $H$ and $K$ be finite groups with $K \subseteq H \subseteq G$. Let $\chi$ be a character of $G$ and $\psi$ be a character of $K$. Then we have*

(a) $\chi \downarrow_H^G \downarrow_K^H = \chi \downarrow_K^G$.

(b) $\psi \uparrow_K^H \uparrow_H^G = \psi \uparrow_K^G$

We close this section with a result called *Mackey's Theorem*. It describes the result of inducing a character followed by restriction.

Let $G$ be a finite group and let $H$ and $K$ be subgroups of $G$. If $\chi$ is a character of $H$ and $s \in G$, we define a character $\chi^s$ of $H^s = sHs^{-1}$ via

$$\chi^s(x) = \chi(s^{-1}xs)$$

for every $x \in H^s$. We have the following theorem.

**Theorem 2.1.34** (Mackey)**.** *With the notation as above, we have*

$$\chi \uparrow_H^G \downarrow_K^G = \sum_{s \in R} \chi^s \downarrow_{H^s \cap K}^{H^s} \uparrow_{H^s \cap K}^{K}$$

*where $R$ is a system of representatives for the double cosets $KgH$ with $g \in G$.*

This theorem will become important when dealing with the generalised symmetric group in Chapter 5.

## 2.2 REPRESENTATION THEORY OF THE SYMMETRIC GROUP

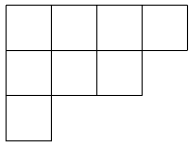
We now focus on the symmetric group $S_n$. Its representation theory over the complex numbers is well-understood. Here, we present results about the irreducible characters of the symmetric group and the decomposition of an important class of characters. For more details, we refer the reader to [45] or [54].

An (integer) *partition* is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ of non-negative integers that sum to a finite number and satisfy $\lambda_i \geqslant \lambda_{i+1}$ for every $i \in \mathbb{N}$. The $\lambda_i$ are called *parts* of the partition $\lambda$. The *size* of $\lambda$ is the number $|\lambda| = \sum_{i \geqslant 1} \lambda_i$ and if $|\lambda| = n$, we say that $\lambda$ is a partition of $n$ and denote this by $\lambda \vdash n$. The largest number $i$ such that $\lambda_i > 0$ is called the *length* of $\lambda$, denoted by $l(\lambda)$.

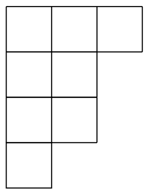
We often write partitions as finite sequences $(\lambda_1, \ldots, \lambda_{l(\lambda)})$ only writing down the non-zero parts. When no confusion can arise, we omit the commas when writing down partitions.

We also write $k^t$ to denote $t$ parts of size $k$, for example $2^1 1^2$ is the sequence $(211)$. The unique partition of 0 will be denoted by $\varnothing$. Furthermore, we denote by Par the set of all integer partitions (of any size) while $\mathrm{Par}_n$ denotes the set of partitions of $n$.

If $\lambda$ is a partition of $n$, then the *Young diagram* of $\lambda$ is an array of $n$ boxes with left-justified rows and top-justified columns such that row $i$ contains exactly $\lambda_i$ boxes. For example, the Young diagram of the partition $(431)$ is:



Each partition gives rise to a *conjugate partition* $\lambda'$ whose parts are the number of boxes in the columns of the Young diagram of $\lambda$ (so the Young diagram of $\lambda'$ is the transpose of the Young diagram of $\lambda$). For example, the conjugate partition of $(431)$ is $(3221)$ and its Young diagram is:



We use two methods for combining partitions. If $\lambda$ and $\mu$ are partitions, then $\lambda \cup \mu$ is the partition of $|\lambda| + |\mu|$ that has as its parts exactly the parts of $\lambda$ and $\mu$, and $\lambda + \mu$ is the componentwise sum, that is $\lambda + \mu = (\lambda_1 + \mu_1, \lambda_2 + \mu_2, \ldots)$. It holds that

$$(\lambda \cup \mu)' = \lambda' + \mu' \quad \text{and} \quad (\lambda + \mu)' = \lambda' \cup \mu'.$$

A partition $\lambda$ is *contained* in a partition $\mu$, denoted by $\lambda \subseteq \mu$, if $\lambda_i \leqslant \mu_i$ for every $i \geqslant 1$. In terms of Young diagrams, this means that the Young diagram of $\lambda$ lies inside the Young diagram of $\mu$ if they are superimposed.

We use the dominance order $\trianglelefteq$ to compare partitions. Let $\lambda$ and $\mu$ be two partitions (not necessarily of the same size). We write $\lambda \trianglelefteq \mu$ and say that $\lambda$ is *dominated* by $\mu$ or $\mu$ *dominates* $\lambda$ if

$$\sum_{i=1}^{k} \lambda_i \leqslant \sum_{i=1}^{k} \mu_i \quad \text{for all } k \geqslant 1.$$

Notice that the dominance order is not a total order. For example, for the partitions $(411)$ and $(33)$, we have neither $(411) \trianglelefteq (33)$ nor $(33) \trianglelefteq (411)$.

A helpful visualisation is the fact that we have $\lambda \trianglelefteq \mu$ if and only if the Young diagram of $\lambda$ can be transformed into the Young diagram of $\mu$ by moving boxes from the end of a row of $\lambda$ to a higher row one at a time (see for example [11, Proposition 2.3]). Figure 2.1 gives an example showing that $(3221) \trianglelefteq (3311)$.

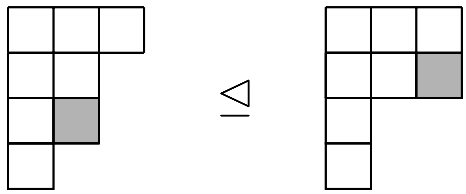


Figure 2.1: The dominance order

When interpreting a partition as a sequence, moving a single box up corresponds to adding vectors of the form $(0, \ldots, 0, 1, 0, \ldots, 0, -1, 0, \ldots)$ to the partition.

The dominance order behaves nicely when considering conjugate partitions.

**Theorem 2.2.1.** *Let $\lambda$ and $\mu$ be two partitions with $|\lambda| = |\mu|$. Then we have*

$$\lambda \trianglelefteq \mu \iff \lambda' \trianglerighteq \mu'.$$

Recall that the number of irreducible characters of a group is equal to the number of conjugacy classes. The conjugacy class of a permutation $\sigma \in S_n$ is determined by its *cycle type*. Every permutation can be written as a product of disjoint cycles which is unique up to order of the cycles. If $\sigma = \sigma_1 \sigma_2 \ldots \sigma_k$ is such a decomposition where the cycles are ordered non-increasingly with respect to their lengths, then the tuple of cycle lengths $(|\sigma_1|, |\sigma_2|, \ldots, |\sigma_k|)$ is called the *cycle type* of $\sigma$. It is a partition of $n$. The cycle type characterises the conjugacy classes of the symmetric group.

**Theorem 2.2.2.** *Two elements $\sigma, \tau \in S_n$ are conjugate if and only if they have the same cycle type. In particular, the number of conjugacy classes of $S_n$ is equal to the number of partitions of $n$.*

We now know that the irreducible characters of the symmetric group can be indexed by the partitions of $n$. It is not immediately obvious how to associate characters with partitions. We start by noticing that a partition $\lambda$ of $n$ defines a subgroup of the symmetric group $S_n$, namely

$$S_\lambda = S_{\lambda_1} \times S_{\lambda_2} \times \ldots \times S_{\lambda_l}$$

where each $S_{\lambda_i}$ permutes a set of $\lambda_i$ elements such that all of these sets are disjoint. Such a subgroup is called a *Young subgroup of shape $\lambda$*. All Young subgroups of the same shape are conjugate to each other. For simplicity, we can assume that $S_{\lambda_1}$ permutes the elements $1, 2, \ldots, \lambda_1$, the group $S_{\lambda_2}$ permutes the elements $\lambda_1 + 1, \ldots, \lambda_1 + \lambda_2$ and so on.

It turns out that we can find all irreducible characters of the symmetric group from the induced characters $\xi^\lambda = 1 \uparrow_{S_\lambda}^{S_n}$. Although these characters are not all irreducible, we can order the partitions of $n$ in such a way that $\xi^{\lambda^{(1)}}$ is irreducible, $\xi^{\lambda^{(2)}}$ contains the character $\xi^{\lambda^{(1)}}$ and a single copy of an irreducible character $\chi^{\lambda^{(2)}}$ and so on. In general, $\xi^{\lambda^{(i)}}$ decomposes into the irreducible characters $\chi^{\lambda^{(k)}}$ for $k < i$ and a unique new irreducible character $\chi^{\lambda^{(i)}}$.

Let us look at the characters $\xi^\lambda$ in more detail. Recall from Theorem 2.1.30 that $\xi^\lambda$ is the permutation character of $S_n$ on the cosets of $S_\lambda$ in $S_n$. A nice way to visualise this action is using $\lambda$-*tableaux* and $\lambda$-*tabloids*.

**Definition 2.2.3.** Let $\lambda$ be a partition of $n$. A $\lambda$-*tableau* is a filling of the Young diagram of $\lambda$ with the numbers $1, \ldots, n$ such that every number appears exactly once.

An *ordered set partition* $P = (P_1, P_2, \ldots)$ of $\{1, 2, \ldots, n\}$ is a tuple of pairwise disjoint subsets $P_i \subseteq \{1, \ldots, n\}$ whose union is $\{1, \ldots, n\}$.

A $\lambda$-*tabloid* is an ordered set partition of the set $\{1, \ldots, n\}$ into subsets such that the $i$-th part has size $\lambda_i$. The underlying partition is called the *shape* of the tableau or tabloid.

For example,

$$\begin{array}{|c|c|c|c|}\hline 6 & 3 & 2 & 7 \\ \hline 5 & 1 & 8 \\ \cline{1-3} 4 \\ \cline{1-1}\end{array}$$

is a (431)-tableau. Forgetting the order of the boxes in a row gives a (431)-tabloid

$$\begin{array}{|cccc|}\hline 6 & 3 & 2 & 7 \\ \hline 5 & 1 & 8 \\ \cline{1-3} 4 \\ \cline{1-1}\end{array} \quad = \quad \begin{array}{|cccc|}\hline 2 & 3 & 6 & 7 \\ \hline 1 & 5 & 8 \\ \cline{1-3} 4 \\ \cline{1-1}\end{array}$$

that represents the ordered set partition $(\{2, 3, 6, 7\}, \{1, 5, 8\}, \{4\})$. We remark that there are $n!$ different $\lambda$-tableaux and $n!/\lambda!$ different $\lambda$-tabloids where $\lambda! = \lambda_1!\lambda_2! \ldots$.

The symmetric group acts on $\lambda$-tableaux and $\lambda$-tabloids in the natural way by applying a permutation to all entries. We see that a Young subgroup $S_\lambda$ stabilises a $\lambda$-tabloid. Hence, the permutation character $\xi^\lambda$ counts the number of fixed points under this action, that is $\xi^\lambda(\sigma)$ is the number of $\lambda$-tabloids fixed by $\sigma$.

**Example 2.2.4.** Let $\lambda = (n)$. There is only one $(n)$-tabloid:

$$\boxed{1 \quad 2 \quad 3 \; \cdots \; n}$$

Hence, we have $\xi^{(n)}(\sigma) = 1$ for every $\sigma \in S_n$ and $\xi^{(n)} = 1_{S_n}$ is the trivial character (which is irreducible).

If $\lambda = (n-1, 1)$, then there are $n$ different $\lambda$-tabloids. Each one is determined completely by the element in its second row and we can associate each $\lambda$-tabloid with a number from 1 to $n$. For example, the (31)-tabloid

$$\begin{array}{|ccc|}\hline 1 & 3 & 4 \\ \hline \multicolumn{1}{|c|}{2} & & \\ \cline{1-1}\end{array}$$

corresponds to the number 2. With this identification, the symmetric group $S_n$ acts on the $(n-1, 1)$-tabloids just like it acts on the set $\{1, \ldots, n\}$ and consequently the permutation characters of these actions are the same. We have seen in Example 2.1.18 that the permutation character of $S_3$ decomposes into the trivial character and another irreducible character. This is true in general for the group $S_n$ for every $n$ although it is not immediately obvious. We have

$$\xi^{(n-1,1)} = \chi^{(n)} + \chi^{(n-1,1)}$$

for every $n \in \mathbb{N}$ with $n \geqslant 2$. It follows that the irreducible character $\chi^{(n-1,1)}$ is the second one in our ordering of the irreducible characters of $S_n$.

If $\lambda = (1, \ldots, 1)$, then there are $n!$ different $\lambda$-tabloids. We find that the stabiliser of a $\lambda$-tabloid is trivial and only the identity fixes a $\lambda$-tabloid. Hence, $\xi^{(1,\ldots,1)}$ is the character of the left regular representation. We have seen in Example 2.1.32 that the character of the regular representation contains every irreducible character of the group. Hence, the irreducible character $\chi^{(1,\ldots,1)}$ of $S_n$ must be the last one in our ordering of the irreducible characters.

We deduce that the first irreducible character of the symmetric group is $\chi^{(n)}$ and the second one is $\chi^{(n-1,1)}$. Continuing this way, we obtain all irreducible characters of the symmetric group. They are given as $\chi^{\lambda}$ where $\lambda$ is a partition of $n$. To avoid confusion, we will from now on use $\lambda$ to index the irreducible characters $\chi^{\lambda}$ and $\mu$ to index the permutation characters $\xi^{\mu}$.

The $\chi^{\lambda}$ correspond to a set of irreducible representations given in terms of *Specht modules*. We will not go into more detail here and refer the reader to [54, Chapter 2.3] instead.

In general, the permutation characters $\xi^{\mu}$ decompose into certain irreducible characters according to the dominance order. This is called *Young's rule*.

**Theorem 2.2.5** (Young's rule)**.** *Let $\mu$ be a partition of $n$. Then there are integers $K_{\lambda\mu}$ such that the decomposition of $\xi^\mu$ in terms of the irreducible characters $\chi^\lambda$ is*

$$\xi^\mu = \sum_{\lambda \trianglerighteq \mu} K_{\lambda\mu}\chi^\lambda. \tag{2.1}$$

*Furthermore, we have $K_{\lambda\lambda} = 1$ for every partition $\lambda$ of $n$.*

**Remark 2.2.6.** Using the inner product of characters, we can restate Young's rule as

$$\langle \xi^\mu, \chi^\lambda \rangle \neq 0 \implies \lambda \trianglerighteq \mu.$$

The converse of this also holds, that is

$$\langle \xi^\mu, \chi^\lambda \rangle \neq 0 \iff \lambda \trianglerighteq \mu. \tag{2.2}$$

This was proven later than Young's rule (2.1). One of the first publications of this result seems to be [40], though the authors state that the result appeared implicitly in other publications already. We will also refer to the stronger version (2.2) as *Young's rule.*

Notice that (2.2) is an equivalence: The set of irreducible constituents of the permutation character $\xi^\mu$ is exactly the set of irreducible characters $\chi^\lambda$ such that $\lambda \trianglerighteq \mu$. We will encounter similar decompositions of permutation characters in the generalised symmetric group in Chapter 5 and even analogous results beyond the realm of permutation characters when dealing with perfect matchings in Chapter 6.

Theorem 2.2.5 and its strengthening determine the irreducible constituents of certain permutation characters. In other words, we know exactly which irreducible characters have a positive multiplicity. However, computing the exact value of the multiplicities, that is computing the coefficients $K_{\lambda\mu}$, is a much harder problem. These numbers are called *Kostka numbers* and have a combinatorial interpretation in terms of semistandard Young tableaux.

A *semistandard* Young tableaux of shape $\lambda$ and content $\mu$ is a filling of the Young diagram of $\lambda$ with positive integers such that the number $i$ appears precisely $\mu_i$ times and such that the numbers in the columns increase strictly and the numbers in the rows are non-decreasing. An example is

| | | | |
|---|---|---|---|
| 1 | 1 | 1 | 4 |
| 2 | 2 | 3 | |
| 3 | | | |

which is a semistandard (431)-tableau with content (3221). The Kostka number $K_{\lambda\mu}$ is the number of semistandard $\lambda$-tableaux with content $\mu$.

For $n = 3$, elementary counting reveals that the Kostka numbers have the following values. The entry in row $\lambda$ and column $\mu$ is the Kostka number $K_{\lambda\mu}$.

| $K_{\lambda\mu}$ | (3) | (21) | (111) |
|---|---|---|---|
| (3) | 1 | 1 | 1 |
| (21) | 0 | 1 | 2 |
| (111) | 0 | 0 | 1 |

Now consider the matrix $K = (K_{\lambda\mu})_{\lambda,\mu\vdash n}$ for an arbitrary value of $n$. Since $K_{\lambda\lambda} = 1$ for every $\lambda \vdash n$ and every $n \in \mathbb{N}$, $K$ has 1s on the diagonal. The ordering of the irreducible characters (or equivalently the partitions of $n$) ensures that $K$ is a triangular matrix. Hence, the determinant of $K$ is 1. It follows that $K$ is invertible and since the entries of $K$ are integers, it follows that the inverse $K^{-1}$ is also an integral matrix.

Applying this insight to Equation (2.1), we obtain the following important statement.

**Theorem 2.2.7.** *Let $\chi^\lambda$ be an irreducible character of the symmetric group and let $\xi^\mu$ be the permutation characters of Young subgroups. There are integers $c_{\lambda\mu}$ such that*

$$\chi^\lambda = \sum_{\mu\vdash n} c_{\lambda\mu}\xi^\mu.$$

*In particular, since the permutation characters $\xi^\mu$ are integer-valued, it follows that the irreducible characters of the symmetric group are integer-valued.*

Theorem 2.2.5 and Theorem 2.2.7 can also be interpreted as saying that the matrix of Kostka numbers is the transition matrix from the basis of irreducible characters to the basis of permutation characters (in the space of class functions). Let us illustrate that explicitly.

**Example 2.2.8.** The character table of the symmetric group $S_3$ was calculated in Example 2.1.26.

| | id | (1,2) | (1,2,3) |
|---|---|---|---|
| $1_{S_3}$ | 1 | 1 | 1 |
| $\chi_{\mathrm{sg}}$ | 1 | −1 | 1 |
| $\chi_2$ | 2 | 0 | −1 |

We have to ensure that the characters of $S_3$ are ordered as $\chi^{(3)}$, $\chi^{(21)}$, $\chi^{(111)}$ so that the indexing agrees with the matrix of Kostka numbers. We already saw that $\chi^{(3)}$ is the trivial character (so the first row of our character table). By Young's rule (2.1), we have

$$\xi^{(21)} = K_{(3)(21)}\chi^{(3)} + K_{(21)(21)}\chi^{(21)} = \chi^{(3)} + \chi^{(21)}.$$

Since $\xi^{(21)}(\mathrm{id}) = 3$ (there are precisely 3 different $(21)$-tabloids), we find that

$$\chi^{(21)}(\mathrm{id}) = \xi^{(21)}(\mathrm{id}) - \chi^{(3)}(\mathrm{id}) = 3 - 1 = 2.$$

So $\chi^{(21)}$ is the character we called $\chi_2$. Finally, the sign character $\chi_{\mathrm{sg}}$ is the character $\chi^{(111)}$. Permuting the characters has the effect of permuting the rows of the character table, so the character table for the ordering $(3)$, $(21)$, $(111)$ is

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & 1 \end{pmatrix}.$$

Now we calculate

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 1 & 0 \\ 6 & 0 & 0 \end{pmatrix}.$$

Each row of the matrix on the right hand side corresponds to one instance of the formula in Theorem 2.2.5 for a particular partition $\mu$. Row 1 corresponds to the permutation character $\xi^{(3)}$ (on $(3)$-tabloids), row 2 corresponds to the permutation character $\xi^{(21)}$ (on $(21)$-tabloids) and row 3 corresponds to the permutation character $\xi^{(111)}$ (on $(111)$-tabloids). The first column counts the number of fixed $\mu$-tabloids of the identity (so, equivalently, the total number of $\mu$-tabloids) and the second and third columns count the number of fixed $\mu$-tabloids of the permutations $(1,2)$ and $(1,2,3)$, respectively.

We can compute the inverse of the matrix of Kostka numbers and find that

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 3 & 1 & 0 \\ 6 & 0 & 0 \end{pmatrix}.$$

Every row of the matrix on the left hand side corresponds to an irreducible character $\chi^{\lambda}$ and hence corresponds to one instance of the formula in Theorem 2.2.7. For example, the sign character $\chi^{(111)}$ satisfies

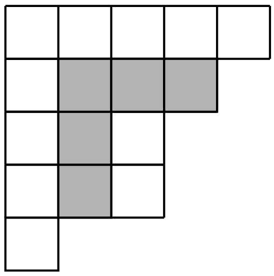
$$\chi^{(111)} = \xi^{(3)} - 2\,\xi^{(21)} + \xi^{(111)}.$$

Importantly, this computation shows that the irreducible characters of the symmetric group have integer values.

The degrees of the irreducible characters of the symmetric group can be calculated via the hook length formula.

**Definition 2.2.9.** Let $\lambda$ be a partition of $n$ and $(i, j)$ be the box in row $i$ and column $j$ of the Young diagram of $\lambda$. The *hook* $H_{i,j}$ is the set of boxes $(i, j')$ with $j' \geqslant j$ and $(i', j)$ with $i' \geqslant i$. The *hook length* $h_{i,j}$ is the number of boxes of $H_{i,j}$.

Consider for example $\lambda = (54331)$. Then the grey boxes of the Young diagram



are the hook $H_{2,2}$ and its hook length is $h_{2,2} = 5$.

We have the following theorem.

**Theorem 2.2.10** (Hook length formula)**.** *Let $\lambda$ be a partition of $n$. Write $(i, j) \in \lambda$ if the Young diagram contains a box in row $i$ and column $j$. Then we have*

$$\deg(\chi^\lambda) = \frac{n!}{\prod_{(i,j)\in\lambda} h_{i,j}}.$$

**Example 2.2.11.** Consider the partition $\lambda = (431)$. We fill the box $(i, j)$ of the Young diagram of $\lambda$ with the hook length $h_{i,j}$:

| | | | |
|---|---|---|---|
| 6 | 4 | 3 | 1 |
| 4 | 2 | 1 | |
| 1 | | | |

Hence, the dimension of the irreducible character $\chi^{(431)}$ is

$$\deg(\chi^{(431)}) = \frac{8!}{6 \cdot 4 \cdot 3 \cdot 1 \cdot 4 \cdot 2 \cdot 1 \cdot 1} = 70.$$

We point out that our overview of the irreducible characters of the symmetric group is far from complete. It is not at all obvious why the irreducible characters $\chi^\lambda$ can be ordered suitably and what the corresponding representations actually look like. Also, proving Young's rule is not as straightforward from the ordering of the irreducible characters of the symmetric group as this treatment may make it look like. We refer the reader to [54, Chapter 2] for a full construction of the irreducible representations of the symmetric group.

The key ideas that will also come up in later chapters are Young subgroups and the decomposition of permutation characters that can be written as the induction of the trivial character from a Young subgroup to the whole group. We will consider interpretations of these characters in terms of suitably defined tableaux and obtain generalisations of Young's rule for them.

# 3

# ASSOCIATION SCHEMES

In this chapter, we give an introduction to the theory of association schemes. We define the basic concepts and give examples to illustrate them. Special attention is paid to conjugacy class schemes which we will be working with in Chapter 5.

The theory of *association schemes* is frequently dubbed "group theory without groups". In fact, Bannai and Ito give this as one possible description of algebraic combinatorics in their famous book "Algebraic Combinatorics I: Association Schemes" [5]. They go on to say that "the concept of association schemes is very fundamental, perhaps the most important, in Algebraic Combinatorics". This is not an overstatement: Association schemes have proven their usefulness by providing a common framework to study codes and designs in many different settings. Once a problem can be stated in terms of association schemes, one has a lot of powerful techniques available to study it, including eigenvalue techniques, linear programming, graph theory and representation theory.

We first provide a brief general introduction to the theory of association schemes, giving examples and basic properties in Section 3.1. In Section 3.2, we will focus on subsets of association schemes that are characterised by their *inner* and *dual distribution*, giving rise to *D-cliques* and *T-designs*. These are the fundamental objects studied in this thesis.

## 3.1 DEFINITIONS, EXAMPLES AND BASIC RESULTS

Here, we collect basic results about association schemes. We refer the reader to [5] and [19] for extensive treatments of the theory. In what follows, the symbol $X$ always denotes a finite set with at least two elements.

**Definition 3.1.1.** An *association scheme* (with $n$ classes) is a pair $(X, \mathcal{R})$ of a finite set $X$ with $|X| \geqslant 2$ and a family of non-empty relations $\mathcal{R} = (R_i)_{i=0,\dots,n}$ subject to the following conditions.

(A0) $\mathcal{R}$ forms a set partition of $X \times X$; that is, $R_i \cap R_j = \varnothing$ for $i \neq j$ and $\bigcup_{i=0}^{n} R_i = X \times X$.

(A1) $R_0$ is the 'identity' relation; that is, $R_0 = \{(x, x) \in X \times X \mid x \in X\}$.

(A2) $\mathcal{R}$ is closed under taking converses; that is,

$$R_i^T = \{(y,x) \in X \times X \mid (x,y) \in R_i\} = R_j$$

for some $j \in \{0,\dots,n\}$.

(A3) Let $i,j,k \in \{0,\dots,n\}$. If $(x,y) \in R_k$, then the number $p_{ij}^k$ of elements $z \in X$ such that $(x,z) \in R_i$ and $(z,y) \in R_j$ only depends on $i$, $j$ and $k$ but not on the choice of $(x,y)$. In other words, we have

$$|\{z \in X \times X \mid (x,z) \in R_i \text{ and } (z,y) \in R_j\}| = p_{ij}^k$$

for every $x,y \in X$ with $(x,y) \in R_k$.

(A4) For every $i,j,k \in \{0,\dots,n\}$ we have $p_{ij}^k = p_{ji}^k$.

The constants $p_{ij}^k$ are called the *intersection numbers* of the association scheme. If $R_i^T = R_j$, then the intersection number $p_{ij}^0$ is called the *valency* of the relation $R_i$. It is denoted by $v_i$ and we find

$$v_i = |\{z \in X \mid (x,z) \in R_i\}|.$$

If $R_i^T = R_i$ for every $i \in \{0,\dots,n\}$, then the association scheme is called *symmetric*.

Before giving further definitions, we give two important examples of association schemes.

**Example 3.1.2.** Let $n,k \in \mathbb{N}$ with $n \geqslant 2k$ and let $X$ be the set of subsets of $\{1,\dots,n\}$ that have precisely $k$ elements. The $i$-th relation $R_i$ is given by

$$R_i = \{(x,y) \in X \times X \mid |x \cap y| = k - i\}.$$

Then $(X,(R_0,\dots,R_k))$ is a symmetric association scheme, called the *Johnson scheme* $J(n,k)$. We have $|X| = \binom{n}{k}$.

The valency of the $i$-th relation is given by

$$v_i = \binom{k}{k-i}\binom{n-k}{i} = \binom{k}{i}\binom{n-k}{i}.$$

The Johnson scheme $J(4,2)$ is depicted in Figure 3.1. The blue edges describe relation $R_0$ (they are the edges of the form $(x,x)$), the olive edges describe relation $R_1$ (they are the edges between sets with exactly 1 element in common) and the orange edges describe relation $R_2$ (they are the edges between disjoint sets). Notice how we can read off the valencies of the scheme from its graph. Every vertex has 1 blue edge, 4 olive edges and 1 orange edge. Hence, the valencies of $J(4,2)$ are $v_0 = 1$, $v_1 = 4$ and $v_2 = 1$.

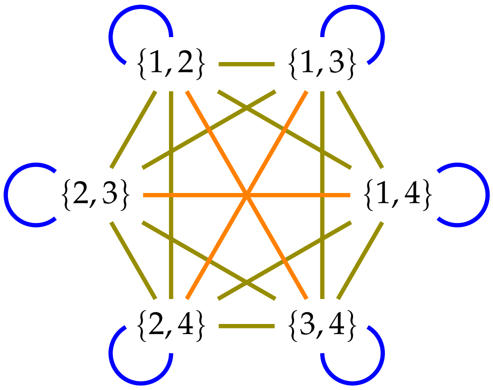



Figure 3.1: The Johnson scheme $J(4,2)$

Any finite group with at least two elements gives rise to an association scheme called the *conjugacy class scheme*. This is the scheme we will be studying to investigate transitivity in the generalised symmetric group.

**Example 3.1.3.** Let $G$ be a finite group with $|G| \geqslant 2$ and denote by $C_0$, $C_1$, $\ldots, C_n$ its conjugacy classes where $C_0 = \{e\}$. The $i$-th relation $R_i$ is given by

$$R_i = \{(g,h) \in G \times G \mid g^{-1}h \in C_i\}.$$

Then $(X, (R_0, \ldots, R_n))$ is an association scheme, called the *conjugacy class scheme*. It is symmetric if and only if each conjugacy class is closed under taking inverses. The valency of the $i$-th relation is given by $v_i = |C_i|$.

A convenient way to work with association schemes is by using matrices. We first introduce some notation.

For finite and non-empty sets $X$ and $Y$, let $\mathbb{C}(X,Y)$ denote the set of all complex $|X| \times |Y|$ matrices with rows indexed by $X$ and columns indexed by $Y$. For a matrix $A \in \mathbb{C}(X,Y)$ and $x \in X$, $y \in Y$, the $(x,y)$-entry of $A$ is denoted by $A(x,y)$. If $|Y| = 1$, then we omit $Y$, so $\mathbb{C}(X)$ is the set of complex column vectors indexed by $X$.

We can describe any relation $R \subseteq X \times X$ using a matrix $A \in \mathbb{C}(X,X)$ given by

$$A(x,y) = \begin{cases} 1, & \text{if } (x,y) \in R, \\ 0, & \text{otherwise.} \end{cases} \tag{3.1}$$

This is called the *adjacency matrix* of the relation.

The following is an equivalent definition of an association scheme using adjacency matrices.

**Definition 3.1.4.** Let $X$ be a finite set with $|X| \geqslant 2$ and let $R_0, \ldots, R_n \subseteq X \times X$ be non-empty relations on $X$. Let $A_i$ be the adjacency matrix of the relation $R_i$.

Then $(X, (R_0, \ldots, R_n))$ is an association scheme if the following conditions are satisfied.

(A0′) $\sum_{i=0}^{n} A_i = J$ where $J$ denotes the all-ones matrix.

(A1′) $A_0 = I$ where $I$ denotes the identity matrix.

(A2′) For every $i \in \{0, \ldots, n\}$ we have $A_i^T = A_j$ for some $j \in \{0, \ldots, n\}$.

(A3′) For every $i, j \in \{0, \ldots, n\}$ we have $A_i A_j = \sum_{k=0}^{n} p_{ij}^k A_k$ for some integers $p_{ij}^k$.

(A4′) For every $i, j \in \{0, \ldots, n\}$ we have $A_i A_j = A_j A_i$.

These conditions are a direct translation of the definition of an association scheme to the language of matrices. It follows from (A3′) that the vector space $\mathbb{A}$ generated by the adjacency matrices of the relations of an association scheme is closed under matrix multiplication. From (A4′), it follows that multiplication in $\mathbb{A}$ is commutative. Finally, it follows from (A0′) that the dimension of $\mathbb{A}$ is $n+1$. This algebra plays a central role in the theory of association schemes.

**Definition 3.1.5.** Let $(X, \mathcal{R})$ be an association scheme and let $A_i$ be the adjacency matrix of the relation $R_i$. The vector space $\mathbb{A} = \mathrm{sp}_{\mathbb{C}}(A_0, \ldots, A_n)$ generated by $A_0, \ldots, A_n$ over the complex numbers is called the *Bose-Mesner algebra* of the association scheme.

Conditions (A1′) and (A2′) imply that $\mathbb{A}$ contains the identity matrix and is closed under conjugate transposition. Thus, since $\mathbb{A}$ is commutative, all matrices in $\mathbb{A}$ are normal and can be simultaneously diagonalised via a unitary matrix $U$.

Now consider a matrix $A \in \mathbb{A}$ with the maximum number of distinct eigenvalues $\lambda_0, \ldots, \lambda_n$ across all matrices in $\mathbb{A}$. Since $\mathbb{A}$ has dimension $n+1$, it is not hard to see that $A$ must in fact have precisely $n+1$ distinct eigenvalues. Then $D = U^* A U$ is a diagonal matrix with the eigenvalues $\lambda_0, \ldots, \lambda_n$ on the diagonal. The $|X| \times |X|$ matrix $D$ defines a set partition $P$ of $X$ where the $k$-th part contains all elements $x \in X$ such that $D(x, x) = \lambda_k$.

Now let $F_k$ be the diagonal matrix given by

$$F_k(x, x) = \begin{cases} 1, & x \in P_k, \\ 0, & \text{otherwise.} \end{cases}$$

Then the matrices $F_k$ are idempotent and satisfy $F_k F_l = 0$ for $k \neq l$ and we can write

$$D = \lambda_0 F_0 + \ldots + \lambda_n F_n.$$

It is easy to see that each $F_k$ can be written as a linear combination of powers of $D$, hence $F_k \in U^*\mathbb{A}U$ for every $k \in \{0, \ldots, n\}$. Thus, it follows that the matrices $E_k = UF_kU^*$ are elements of $\mathbb{A}$ and are pairwise orthogonal idempotents. Therefore, there exists a basis $E_0, \ldots, E_n$ of $\mathbb{A}$ consisting of Hermitian matrices with the property

$$E_k E_l = \delta_{kl} E_k \tag{3.2}$$

for every $k, l \in \{0, \ldots, n\}$. The matrices $E_k$ are the *minimal idempotents* of $\mathbb{A}$.

We collect some properties of the minimal idempotents in the following theorem.

**Theorem 3.1.6.** *Let $\mathbb{A}$ be the Bose-Mesner algebra of an association scheme $(X, \mathcal{R})$ with $n$ classes. Then there exists a unique basis $E_0, \ldots, E_n$ of $\mathcal{A}$ consisting of pairwise orthogonal idempotent matrices that are Hermitian. Moreover, they satisfy*

$$\sum_{k=0}^{n} E_k = I \quad \text{and} \quad \frac{1}{|X|} J = E_k \text{ for some } k \in \{0, \ldots, n\}$$

*where $I$ is the $|X| \times |X|$ identity matrix and $J$ is the $|X| \times |X|$ matrix whose entries are all equal to* 1.

Without loss of generality, we can assume that $E_0 = \frac{1}{|X|}J$. Notice that since the matrices $E_k$ are Hermitian and idempotent, they are positive semidefinite. If $x \in \mathbb{C}^{|X|}$, then we have

$$x^* E_k x = x^* E_k E_k x = x^* E_k^* E_k x = (E_k x)^* (E_k x) \geqslant 0.$$

Notice that there are many striking similarities between the canonical bases $A_0, \ldots, A_n$ and $E_0, \ldots, E_n$. Table 3.1 shows some of their properties. Here, we denote by $\circ$ the *Hadamard product* of matrices, that is the componentwise multiplication. The numbers $q_{ij}^k$ are called *Krein parameters* of the association scheme.

| $A_i$ | $E_k$ |
|---|---|
| $A_0 = I$ | $E_0 = \frac{1}{\lvert X\rvert}J$ |
| $\sum_{i=0}^{n} A_i = J$ | $\sum_{k=0}^{n} E_k = I$ |
| $A_i \circ A_j = \delta_{ij} A_i$ | $E_k E_l = \delta_{kl} E_k$ |
| $A_i A_j = \sum_{k=0}^{n} p_{ij}^k A_k$ | $E_i \circ E_j = \sum_{k=0}^{n} q_{ij}^k E_k$ |

Table 3.1: Properties of the adjacency matrices and the minimal idempotents

Let $V_k$ be the column space of $E_k$. From (3.2), the spaces $V_k$ are pairwise orthogonal and

$$\mathbb{C}(X) = \bigoplus_{k=0}^{n} V_k. \tag{3.3}$$

The spaces $V_k$ are called the *eigenspaces* of the association scheme because they are the common eigenspaces of the matrices $A_i$. The numbers $m_k = \dim(V_k) = \operatorname{rank}(E_k)$ are called the *multiplicities* of the association scheme. We also note that the idempotent $E_k$ is the orthogonal projection onto $V_k$.

Since both $A_0, \ldots, A_n$ and $E_0, \ldots, E_n$ are bases of $\mathbb{A}$, we can transform one into the other using a linear transformation. Therefore, there exist complex numbers $P_i(k)$ and $Q_k(i)$ ($k, i \in \{0, \ldots, n\}$) such that

$$A_i = \sum_{k=0}^{n} P_i(k) E_k \tag{3.4}$$

and

$$E_k = \frac{1}{|X|} \sum_{i=0}^{n} Q_k(i) A_i. \tag{3.5}$$

The numbers $P_i(k)$ are called *eigenvalues* of the association scheme and the numbers $Q_k(i)$ are called *dual eigenvalues* of the association scheme. We will later see that these numbers are of utmost importance in the coming chapters, especially when dealing with designs.

The justification for the name *eigenvalues* is the following. Multiplying (3.4) by $E_k$, we obtain

$$A_i E_k = P_i(k) E_k.$$

Hence, $P_i(k)$ is an eigenvalue of $A_i$ and every column of $E_k$ is an eigenvector. Similarly, we can multiply (3.5) componentwise by $A_i$ and obtain

$$A_i \circ E_k = \frac{1}{|X|} Q_k(i) A_i.$$

Thus, the entries of the matrix $E_k$ are precisely the dual eigenvalues $Q_k(i)$ divided by $|X|$.

We collect the eigenvalues in the *P-matrix* $P = (P_i(j))_{i,j=0,\ldots,n}$ and the dual eigenvalues in the *Q-matrix* $Q = (Q_i(j))_{i,j=0,\ldots,n}$

The following theorem gives some properties of the (dual) eigenvalues.

**Theorem 3.1.7.** *Let $(X, \mathcal{R})$ be an association scheme and let $P_i(k)$ and $Q_k(i)$ be its (dual) eigenvalues. Then we have* (*a*) $P_0(k) = 1$ *for every* $k$,

(*b*) $Q_0(i) = 1$ *for every* $i$,

(*c*) $P_i(0) = v_i$ *for every* $i$,

(*d*) $Q_k(0) = m_k$ *for every* $k$,

(*e*) *The* $P$*-matrix and the* $Q$*-matrix satisfy* $PQ = |X|I$.

Notice that for a symmetric association scheme, the matrices $A_i$ are symmetric and thus have real eigenvalues. Hence, by Theorem 3.1.7, the dual eigenvalues of a symmetric association scheme are also real.

**Example 3.1.8.** Consider the Johnson scheme $J(4,2)$. We order the subsets in the following way:

$$\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}$$

The adjacency matrices of the Johnson scheme $J(4,2)$ are given by

$$A_0 = \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0&1&1&1&1&0\\ 1&0&1&1&0&1\\ 1&1&0&0&1&1\\ 1&1&0&0&1&1\\ 1&0&1&1&0&1\\ 0&1&1&1&1&0 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0&0&0&0&0&1\\ 0&0&0&0&1&0\\ 0&0&0&1&0&0\\ 0&0&1&0&0&0\\ 0&1&0&0&0&0\\ 1&0&0&0&0&0 \end{pmatrix}.$$

The eigenvalues of $A_0$ are 1, 1, 1, 1, 1, 1, the eigenvalues of $A_1$ are 4, $-2$, $-2$, 0, 0, 0 and the eigenvalues of $A_2$ are 1, 1, 1, $-1$, $-1$, $-1$. The decomposition of $\mathbb{C}^6$ into common eigenspaces is

$$\mathbb{C}^6 = \left\langle \begin{pmatrix}1\\1\\1\\1\\1\\1\end{pmatrix} \right\rangle \oplus \left\langle \begin{pmatrix}1\\0\\-1\\-1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\-1\\-1\\1\\0\end{pmatrix} \right\rangle \oplus \left\langle \begin{pmatrix}1\\0\\0\\0\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\\-1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\-1\\0\\0\end{pmatrix} \right\rangle.$$

Thus, we find

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 4 & -2 & 0 \\ 1 & 1 & -1 \end{pmatrix}.$$

We get the dual eigenvalues by computing

$$Q = |X|P^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 2 \\ 3 & 0 & -3 \end{pmatrix}.$$

Notice that all entries in the first row are equal to 1 and the entries in the first column are the dimensions of the common eigenspaces.

The minimal idempotents can now easily be obtained from the dual eigenvalues. Using (3.5), for $(x, y) \in R_i$ we have

$$E_k(x, y) = \frac{1}{|X|} Q_k(i). \tag{3.6}$$

We find

$$E_0 = \frac{1}{6}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \quad E_1 = \frac{1}{6}\begin{pmatrix} 2 & -1 & -1 & -1 & -1 & 2 \\ -1 & 2 & -1 & -1 & 2 & -1 \\ -1 & -1 & 2 & 2 & -1 & -1 \\ -1 & -1 & 2 & 2 & -1 & -1 \\ -1 & 2 & -1 & -1 & 2 & -1 \\ 2 & -1 & -1 & -1 & -1 & 2 \end{pmatrix},$$

$$E_2 = \frac{1}{6}\begin{pmatrix} 3 & 0 & 0 & 0 & 0 & -3 \\ 0 & 3 & 0 & 0 & -3 & 0 \\ 0 & 0 & 3 & -3 & 0 & 0 \\ 0 & 0 & -3 & 3 & 0 & 0 \\ 0 & -3 & 0 & 0 & 3 & 0 \\ -3 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}.$$

Notice that the traces of these matrices equal their rank and thus are equal to the multiplicities $m_k$ because the matrices are idempotent.

We now focus on the conjugacy class scheme. Its eigenvalues and dual eigenvalues can be given in terms of irreducible characters.

**Theorem 3.1.9.** *Let $G$ be a finite group with conjugacy classes $C_0 = \{e\}, C_1, \ldots, C_n$ and irreducible characters $\chi_0, \chi_1, \ldots, \chi_n$. Denote by $\chi_k(C_i)$ the value of the character $\chi_k$ on the conjugacy class $C_i$. Then the eigenvalues are given by*

$$P_i(k) = \frac{|C_i|}{\deg(\chi_k)} \overline{\chi_k(C_i)}$$

*and the dual eigenvalues are given by*

$$Q_k(i) = \deg(\chi_k)\, \chi_k(C_i).$$

**Example 3.1.10.** Let us come back to the symmetric group $S_3$ with conjugacy classes $C_0 = \{\mathrm{id}\}$, $C_1 = \{(1,2),(1,3),(2,3)\}$ and $C_2 = \{(1,2,3),(1,3,2)\}$. We have seen in Example 2.1.26 that the character table of $S_3$ is given by

| | $C_0$ | $C_1$ | $C_2$ |
|---|---|---|---|
| $\chi^{(3)}$ | 1 | 1 | 1 |
| $\chi^{(21)}$ | 2 | 0 | $-1$ |
| $\chi^{(111)}$ | 1 | $-1$ | 1 |

where we use the standard indexing of the irreducible characters of the symmetric group using partitions. From Theorem 3.1.9, we find that the character table is *almost* the $Q$-matrix. We just have to multiply the row indexed by $\chi^\lambda$ by $\deg(\chi^\lambda)$ which is equal to $\chi^\lambda(C_0)$ and obtain

$$Q = \begin{pmatrix} 1 & 1 & 1 \\ 4 & 0 & -2 \\ 1 & -1 & 1 \end{pmatrix}.$$

We can now either invert $Q$ to obtain the eigenvalues or use the formula from Theorem 3.1.9.

## 3.2 CODES AND DESIGNS IN ASSOCIATION SCHEMES

We now investigate subsets of association schemes. The key observation is that objects that have nice combinatorial properties can often be characterised using properties of an association scheme. In his thesis from 1973 [19], Delsarte showed how well-studied objects like error-correcting codes can be characterised as special subsets of an association scheme, especially the Hamming scheme and the Johnson scheme. He called these subsets *cliques* and *designs* and demonstrated that these definitions also yield interesting combinatorial structures in other association schemes. In fact, in their paper from 1998 [22], Delsarte and Levenshtein point out that "in coding theory and related subjects, an association scheme [...] should mainly be viewed as a "structured space" in which the objects of interest (such as codes, or designs) are living".

We now give the relevant definitions that describe these objects of interest. In this section, $(X, \mathcal{R})$ is always an association scheme with $n$ classes.

**Definition 3.2.1.** Let $Y \subseteq X$ be a non-empty subset. The *inner distribution* of $Y$ is the tuple $a = (a_0, \ldots, a_n)$, where

$$a_i = \frac{1}{|Y|} \sum_{x,y \in Y} A_i(x, y). \tag{3.7}$$

For a subset $Y \subseteq X$, we denote by $\mathbb{1}_Y \in \mathbb{C}(X)$ the *characteristic vector* of $Y$, that is

$$\mathbb{1}_Y(x) = \begin{cases} 1, & x \in Y, \\ 0, & \text{otherwise.} \end{cases}$$

With this notation, the inner distribution of a subset $Y$ can also be written as

$$a_i = \frac{1}{|Y|} \mathbb{1}_Y^T A_i \mathbb{1}_Y = \frac{|(Y \times Y) \cap R_i|}{|Y|}.$$

The following properties of the inner distribution are immediate.

**Lemma 3.2.2.** *Let $Y \subseteq X$ and $(a_0, \ldots, a_n)$ be its inner distribution. Then:*

(*a*) $a_i \geqslant 0$ *for every* $i \in \{0, \ldots, n\}$

(*b*) $a_0 = 1$

(*c*) $a_0 + \ldots + a_n = |Y|$

For an element $x \in X$, the entry $a_i$ is the average number of elements $y \in Y$ such that $(x, y) \in R_i$. In particular, we have $a_i = 0$ if and only if no elements of $Y$ are in relation $i$ to each other. This interpretation is particularly useful if the relations are defined by a metric.

The inner distribution has a nice interpretation in terms of coding theory and the Hamming metric. Let us define these notions.

Let $A$ be a non-empty finite set, called an *alphabet*. An element of $A^n$ is called a word over $A$ of length $n$. We can measure the distance between two words $x, y \in A^n$ via

$$d(x, y) = |\{i \in \{1, \ldots, n\} \mid x_i \neq y_i\}|.$$

Then $d : A^n \times A^n \to \{0, \ldots, n\}$ is a metric, called the *Hamming metric*. An error-correcting code $C$ of length $n$ over $A$ is a non-empty subset of $A^n$. The elements of $C$ are called *codewords*. The code $C$ has *minimum distance d* if we have $d(x,y) \geqslant d$ for any two distinct codewords $x, y \in C$. Remembering our intuition about the word *code* from the introduction, we see that an error-correcting code is a set of words that are 'far apart' in the Hamming metric (that is, any two distinct codewords differ in at least $d$ positions). For a more detailed introduction to coding theory, we refer the reader to [44].

The idea is the following: For every codeword $c \in C$, we can consider the ball of radius $r$ in the Hamming metric defined by

$$B_r(c) = \{x \in A^n \mid d(x,c) \leqslant r\}.$$

If an error-correcting code $C$ has minimum distance $d$, then the balls $B_r(c)$ of radius $r \leqslant (d-1)/2$ are disjoint for distinct codewords. Hence, if at most $(d-1)/2$ entries of a codeword are altered, then we can uniquely identify the original codeword.

We can give a direct interpretation of error-correcting codes and the inner distribution using the *Hamming scheme*.

**Example 3.2.3.** Let $A$ be an alphabet of size $q$ (not necessarily a prime power) and $n \in \mathbb{N}$. The *Hamming scheme* is the association scheme $H(n,q) = (A^n, (R_0, \ldots, R_n))$ where the $i$-th relation is given by

$$R_i = \{(x,y) \in A^n \times A^n \mid d(x,y) = i\}.$$

If $q$ is a prime power, we assume that $A = \mathbb{F}_q$ is the finite field of order $q$. A nice way to visualise the binary Hamming scheme is using the $n$-dimensional hypercube. Its vertices are the elements of $\{0,1\}^n$ and two vertices are joined by an edge if and only if their Hamming distance is 1.

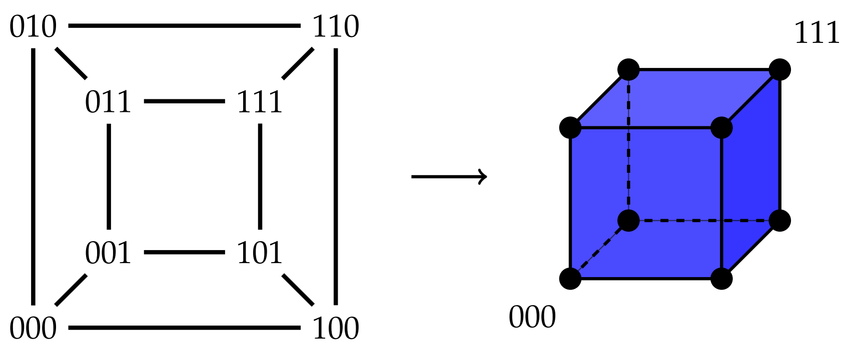



Figure 3.2: The Hamming Scheme $H(3,2)$

Two elements $x, y \in \{0,1\}^n$ have Hamming distance $k$ if and only if the shortest path from $x$ to $y$ in the hypercube has $k$ edges.

An error-correcting code $C$ over the alphabet $A$ with length $n$ and minimum distance $d$ is a subset of the Hamming scheme and its inner distribution is

$$a = (1, 0, \dots, 0, a_d, a_{d+1}, \dots, a_n)$$

for some non-negative integers $a_i$. Note that $a_i = 0$ if and only if no two elements $x, y \in C$ are in relation $i$ to each other. Since the $i$-th relation of the Hamming scheme is given by Hamming distance $i$, any subset of $A^n$ that has the above inner distribution is an error-correcting code of minimum distance $d' \geqslant d$. Note that we do not assume that $a_d \neq 0$.

This suggests the following definition.

**Definition 3.2.4.** Let $D \subseteq \{1, \dots, n\}$ and $Y$ be a non-empty subset of $X$. Then $Y$ is called a *D-clique* if

$$a_i = 0 \text{ for every } i \in \{1, \dots, n\} \setminus D.$$

It is a convention to denote by $D$ the subset of positions where the inner distribution can be non-zero. We do *not* assume that $a_i \neq 0$ for every $i \in D$. A $D$-clique is also called a *D-code*, especially in association schemes that come from a metric.

**Example 3.2.5.** An error-correcting code over $\mathbb{F}_q$ of length $n$ and minimum distance $d$ is a $\{d, d+1, \dots, n\}$-clique in $H(n, q)$. For example for $q = 2$, the set $C$ of solutions to the system of linear equations $Ax = 0$ where

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

is an error-correcting code of length 7 and minimum distance 3. It belongs to the family of *Hamming codes* and we have

$$\begin{aligned} C = \{ & 0000000, 1100001, 1010010, 1001100, \\ & 1111111, 0011110, 0101101, 0110011, \\ & 0110100, 0101010, 0011001, 0000111, \\ & 1001011, 1010101, 1100110, 1111000\}. \end{aligned}$$

The inner distribution of $C$ in the scheme $H(3, 2)$ is

$$a = (1, 0, 0, 7, 7, 0, 0, 1).$$

Hence, $C$ is a $\{3, 4, 5, 6, 7\}$-clique. It is also a $D$-clique for every $D \supseteq \{3, 4, 7\}$. From the inner distribution, we can deduce that any two distinct codewords have Hamming distance 3, 4 or 7.

The inner distribution has a nice interpretation when the relations of an association scheme are given in terms of a metric. This is not always the case but we will see that the definition of a $D$-clique also gives interesting combinatorial structures in other association schemes.

Having found a way to study codes in the language of association schemes, we now seek a way to study designs in the language of association schemes. We need the following definition.

**Definition 3.2.6.** Let $Y \subseteq X$ be a non-empty subset. The *dual distribution* of $Y$ is the tuple $a' = (a'_0, \ldots, a'_n)$, where

$$a'_k = \frac{|X|}{|Y|} \sum_{x,y \in Y} E_k(x, y). \tag{3.8}$$

Using characteristic vectors and writing the idempotents in terms of the adjacency matrices according to (3.5), we arrive at the following alternative way to compute the dual distribution of a subset $Y$.

$$a'_k = \frac{|X|}{|Y|} \mathbb{1}_Y^T E_k \mathbb{1}_Y = \sum_{i=0}^{n} Q_k(i) a_i. \tag{3.9}$$

Hence, the dual distribution is a linear transform of the inner distribution, also called the $Q$-transform of the inner distribution. Thus, knowledge of the dual eigenvalues $Q_k(i)$ is crucial for computing the dual distribution.

The dual distribution has properties similar to the inner distribution.

**Lemma 3.2.7.** *Let $Y \subseteq X$ and $(a'_0, \ldots, a'_n)$ be its dual distribution. Then:*

(*a*) $a'_i \geqslant 0$ *for every* $i \in \{0, \ldots, n\}$

(*b*) $a'_0 = |Y|$

(*c*) $a'_0 + \ldots + a'_n = |X|$

We give a proof of the non-negativity of the dual distribution because it also gives insight into when an entry is equal to 0. Recall that the matrices $E_k$ are Hermitian and calculate

$$\frac{|Y|}{|X|} a'_k = \mathbb{1}_Y^T E_k \mathbb{1}_Y = \mathbb{1}_Y^* E_k^* E_k \mathbb{1}_Y = ||E_k \mathbb{1}_Y||^2. \tag{3.10}$$

Recall that the columns of $E_k$ span the eigenspace $V_k$. Since

$$E_k \mathbb{1}_Y = (\mathbb{1}_Y^* E_k^*)^* = (\mathbb{1}_Y^T E_k)^*,$$

it follows from (3.10) that the case $a'_k = 0$ occurs if and only if $\mathbb{1}_Y$ is orthogonal to $V_k$.

It turns out that just like error-correcting codes have zeroes in their inner distribution in the Hamming scheme, block designs have zeroes in their dual distribution in the Johnson scheme. However, this is not as straightforward to see as it is for codes in the Hamming scheme.

**Example 3.2.8.** Consider the Fano plane from Figure 1.1. It is a 2-$(7,3,1)$ design with blocks

$$123, 147, 156, 246, 257, 345, 367.$$

We can interpret the blocks as a subset $Y$ of the Johnson scheme $J(7,3)$. The inner distribution of $Y$ is given by

$$a = (1,0,6,0).$$

Any two distinct blocks of the Fano plane intersect in a unique point so the inner distribution only has two non-zero entries.

The dual eigenvalues can be computed from the adjacency matrices of the Johnson scheme but as they are of size $35 \times 35$, we will not present the calculation here. Instead, we use an explicit formula for the eigenvalues of the Johnson scheme (see [19, Theorem 4.6]), given in terms of Eberlein polynomials. We find that the $Q$-matrix of $J(7,3)$ is

$$Q = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 6 & \frac{5}{2} & -1 & -\frac{9}{2} \\ 14 & 0 & -\frac{7}{3} & 7 \\ 14 & -\frac{7}{2} & \frac{7}{3} & -\frac{7}{2} \end{pmatrix}.$$

Thus, the dual distribution of $Y$ is

$$a' = Qa = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 6 & \frac{5}{2} & -1 & -\frac{9}{2} \\ 14 & 0 & -\frac{7}{3} & 7 \\ 14 & -\frac{7}{2} & \frac{7}{3} & -\frac{7}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 6 \\ 0 \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \\ 0 \\ 28 \end{pmatrix}.$$

Notice that $a'_1 = a'_2 = 0$.

This suggests the following definition.

**Definition 3.2.9.** Let $T \subseteq \{1, \ldots, n\}$ and $Y$ be a non-empty subset of $X$. Then $Y$ is called a *$T$-design* if

$$a'_k = 0 \text{ for every } k \in T.$$

Note that we do not require that $a'_k \neq 0$ for $k \in \{1,\ldots,n\}\backslash T$. It follows that a $T$-design is also a $T'$-design for every subset $T' \subseteq T$.

Using this language, the Fano plane is a $\{1,2\}$-design in the Johnson scheme $J(7,3)$ (and it is also a $\{1\}$-design and a $\{2\}$-design).

While the inner distribution is defined in terms of counting pairs which gives a combinatorial interpretation of the entries, there is no immediate combinatorial interpretation of the entries of the dual distribution. Zeroes in the dual distribution of a set $Y$ correspond to orthogonality of the characteristic vector $\mathbb{1}_Y$ to some eigenspaces, but the actual combinatorial interpretation of $T$-designs varies greatly.

Let us illustrate this through some examples. First, we give a characterisation of designs in the Johnson scheme.

**Theorem 3.2.10.** *Let $Y$ be a subset of the Johnson scheme $J(n,k)$ with dual distribution $a' = (a'_0,\ldots,a'_k)$. Then $Y$ is a $\{1,\ldots,t\}$-design if and only if $Y$ is a $t$-$(n,k,\lambda)$ block design for some constant $\lambda$.*

*Proof.* We give a proof based on a proof by Delsarte [19, Theorem 4.7]. It uses incidence matrices that we will also encounter in Chapter 5.

Let $M_t$ be the incidence matrix between $t$-subsets and $k$-subsets of $\{1,\ldots,n\}$. The entry indexed by a $t$-set $A$ and a $k$-set $B$ is given by

$$M_t(A,B) = \begin{cases} 1, & \text{if } A \subseteq B, \\ 0, & \text{otherwise.} \end{cases}$$

Then a set $Y$ consisting of $k$-subsets of $\{1,\ldots,n\}$ is a $t$-$(n,k,\lambda)$ block design if and only if

$$M_t \cdot \mathbb{1}_Y = \lambda \cdot \mathbb{1}_t \tag{3.11}$$

where $\mathbb{1}_t$ denotes the all-ones vector indexed by all $t$-subsets of $\{1,\ldots,n\}$. Every row of the matrix product $M_t \cdot \mathbb{1}_Y$ is indexed by a $t$-subset $A$ and evaluates to the number of subsets in $Y$ that contain $A$. Hence, $Y$ is a $t$-design if and only if $M_t \cdot \mathbb{1}_Y$ is a multiple of the all-ones vector.

For the all-ones vector $\mathbb{1}_k$ indexed by all $k$-subsets of $\{1,\ldots,n\}$, we have $M_t \cdot \mathbb{1}_k = \binom{n-t}{k-t} \cdot \mathbb{1}_t$. Hence, we can rewrite Equation (3.11) as

$$M_t \cdot \left( \mathbb{1}_Y - \frac{\lambda}{\binom{n-t}{k-t}} \cdot \mathbb{1}_k \right) = 0. \tag{3.12}$$

Now consider the matrix $C_t = M_t^T M_t$. Then $\ker(C_t) = \ker(M_t)$, so Equation (3.12) is equivalent to

$$C_t \cdot \left( \mathbb{1}_Y - \frac{\lambda}{\binom{n-t}{k-t}} \cdot \mathbb{1}_k \right) = 0. \tag{3.13}$$

Delsarte shows that the matrix $C_t$ is a linear combination of the orthogonal idempotents $E_0, \ldots, E_t$ with non-zero coefficients. Hence, Equation (3.13) holds if and only if

$$E_i \cdot \left( \mathbb{1}_Y - \frac{\lambda}{\binom{n-t}{k-t}} \cdot \mathbb{1}_k \right) = 0 \quad \text{for every } i \in \{0, \ldots, t\}. \tag{3.14}$$

For $i > 0$, we have $E_i \cdot \mathbb{1}_k = 0$, so Equation (3.14) reduces to $E_i \cdot \mathbb{1}_Y = 0$. From (3.10), we find that this is equivalent to $a'_i = 0$. For $i = 0$, we have $E_0 = \binom{n}{k}^{-1} J$ where $J$ denotes the all-ones matrix and the equation reduces to a well-known identity on the parameters of block designs.

Hence, $Y$ is a $t$-design if and only if $a'_1 = \ldots = a'_t = 0$, hence if and only if it is a $\{1, \ldots, t\}$-design. □

Using Theorem 3.2.10, we could have deduced the dual distribution of the Fano plane without computing the dual eigenvalues. Since the Fano plane is a 2-$(7, 3, 1)$ block design, its dual distribution must satisfy $a'_1 = a'_2 = 0$. Furthermore, we know that $a'_0 = |Y|$ and $a'_0 + a'_1 + a'_2 + a'_3 = |X|$ by Lemma 3.2.7. As the Fano plane has 7 blocks and the base set of the Johnson scheme $J(7, 3)$ has 35 elements, we deduce that the dual distribution must be $a' = (7, 0, 0, 28)$.

Next, we give a characterisation of designs in the Hamming scheme. Before we can do so, we need a definition.

**Definition 3.2.11.** Let $A$ be an alphabet of size $q$ (not necessarily a prime power) and let $t, n \in \mathbb{N}$ with $t \leqslant n$. An *orthogonal array* of strength $t$ is a non-empty subset $Y$ of $A^n$ (which we interpret as an array with $|Y|$ rows and $n$ columns) such that the restriction of $Y$ to any $t$ columns contains every element of $A^t$ exactly $|Y|/|A|^t$ times. The number $|Y|/|A|^t$ is called the *index* of the orthogonal array.

**Example 3.2.12.** The matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

is an orthogonal array of strength 2. Every two columns contain the elements 00, 01, 10 and 11 precisely once.

In general, orthogonal arrays are $T$-designs in the Hamming scheme.

**Theorem 3.2.13.** *Let $Y$ be a subset of the Hamming scheme $H(n, q)$ with dual distribution $a' = (a'_0, \ldots, a'_n)$. Then $Y$ is a $\{1, \ldots, t\}$-design if and only if $Y$ is an orthogonal array of strength $t$ with $|Y|$ rows and $n$ columns.*

Notice that this once again agrees with our intuition that a design in an association scheme is a structure with a property of 'even coverage'. An orthogonal array covers each possible $t$-tuple in any set of $t$ coordinates constantly often.

**Example 3.2.14.** Coming back to the previous example, we can compute the inner and dual distribution. Let $Y = \{000, 011, 101, 110\}$, then $Y$ is a subset of the Hamming scheme $H(3,2)$. The inner distribution of $Y$ is

$$a = (1, 0, 3, 0)$$

and the dual distribution of $Y$ is

$$a' = (4, 0, 0, 4).$$

Calling a $\{d, d+1, \dots, n\}$-code a $d$-code and a $\{1, \dots, t\}$-design a $t$-design in the Hamming scheme, we see that $Y$ is both a 2-code and a 2-design.

Furthermore, we remark that the Hamming code from Example 3.2.5 is both a 3-code and a 3-design. The 16 codewords form an orthogonal array of strength 3. It follows that its dual distribution satisfies $a'_1 = a'_2 = a'_3 = 0$.

We will see in Section 5.6 that subsets which are both codes and designs are very special and their distributions are sometimes completely determined.

We will encounter orthogonal arrays again when we construct designs in the generalised symmetric group (see Proposition 5.7.1).

One can also give an interpretation of $T$-designs in the conjugacy class scheme of the symmetric group. However, we will not do so immediately but do so in the next chapter. We give the characterisation in more detail because the techniques used as well as the results can be thought of as a template for the investigation in later chapters.

For future reference, we state a formula for the dual distribution in the conjugacy class scheme of an arbitrary group $G$ that is easily obtained from Theorem 3.1.9. Denote by $\chi_0, \dots, \chi_n$ the irreducible characters of $G$. Then the $k$-th entry of the dual distribution $a'$ of a non-empty subset $Y \subseteq G$ is given by

$$a'_k = \frac{\deg(\chi_k)}{|Y|} \sum_{x,y \in Y} \chi_k(x^{-1}y). \tag{3.15}$$

We remark that the definition of $T$-designs yields very different combinatorial structures depending on the specific association scheme. Both block designs and orthogonal arrays are versatile structures with many applications that have been (and still are) studied extensively. In the language of association schemes, they are the same thing: $T$-designs where $T = \{1, \dots, t\}$. It is our belief that the definition of $T$-designs in association schemes both captures important already existing combinatorial structures and at the same time gives rise to new ones. While there is no general combinatorial interpretation of $T$-designs in association schemes, the following quote from Delsarte's thesis [19, p. 33] describes the situation perfectly.

> "In general, we can give no clear "combinatorial" interpretation for the concept of a $T$-design. However, as we shall see [...], some $T$-designs in the Hamming and Johnson scheme are among the most classical combinatorial configurations. This motivates the present general definition, the "conjecture" being that $T$-designs will often have interesting properties."

Motivated by this, our main focus in this thesis are $T$-designs in the generalised symmetric group and the perfect matching association scheme.

The following is a comparison between the inner and the dual distribution, highlighting their similarities and the notion of duality in association schemes.

| inner distribution | dual distribution |
|---|---|
| $a_i = \frac{1}{\lvert Y \rvert} \mathbb{1}_Y^T A_i \mathbb{1}_Y$ | $a'_k = \frac{\lvert X \rvert}{\lvert Y \rvert} \mathbb{1}_Y^T E_k \mathbb{1}_Y$ |
| $a_i \geqslant 0$ | $a'_k \geqslant 0$ |
| $a_0 = 1$ | $a'_0 = \lvert Y \rvert$ |
| $a_0 + \ldots + a_n = \lvert Y \rvert$ | $a'_0 + \ldots + a'_n = \lvert X \rvert$ |
| zeroes in $a$ define $D$-code | zeroes in $a'$ define $T$-design |

Table 3.2: Properties of the inner and the dual distribution

## 3.3 LINEAR PROGRAMMING

We end this chapter with a short introduction to linear programming. Linear programming yields powerful methods to obtain bounds on the sizes of codes and designs in association schemes. These methods were introduced by Delsarte [19]. We have seen that the structure of an association scheme, specifically the inner distribution and the dual distribution, provides a useful framework to study codes and designs. The fact that both distributions are non-negative and linear transforms of each other is what enables the use of linear programming.

In coding theory, the central topic is balancing good error-correction and being able to transmit a lot of data. A higher minimum distance of a code yields better error-correction properties but it also limits how big a code can be. Hence, we are interested in upper bounds on the size of codes in terms of the minimum distance. We first introduce some terminology of linear programming and then present Delsarte's famous linear programming method. We refer the reader to [55] and [64] for an extensive introduction to linear programming.

**Definition 3.3.1.** Let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$. Then the *primal linear program* (primal LP) is the problem of finding a vector $x \in \mathbb{R}^n$ such that $c^T x$ is maximal under the constraints $Ax \leqslant b$ and $x \geqslant 0$. We write this linear program in the following way:

$$\begin{aligned} \max_{x\in\mathbb{R}^n} \quad & c^T x \\ \text{s.t.} \quad & x \geqslant 0 \\ & Ax \leqslant b \end{aligned} \tag{3.16}$$

Here, $x \geqslant 0$ means $x_i \geqslant 0$ for every $i \in \{1,\dots,n\}$. The map $x \mapsto c^T x$ is called the *objective function*. A vector $x \in \mathbb{R}^n$ satisfying the constraints $x \geqslant 0$ and $Ax \leqslant b$ is called a *feasible solution*. The linear program is called *bounded* if $c^T x$ is bounded over all feasible solutions $x$. Otherwise, the linear program is called *unbounded*. If the linear program is bounded, then a feasible solution $x^*$ is called *optimal* if $c^T x \leqslant c^T x^*$ for every feasible solution $x$. The value $c^T x^*$ for an optimal solution $x^*$ is called the *optimum* of the linear program.

Every primal linear program has a *dual*. Let again $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$. Then the *dual linear program* (dual LP) is the problem of finding a vector $y \in \mathbb{R}^m$ such that $b^T y$ is minimal under the constraints $A^T y \geqslant c$ and $y \geqslant 0$. We write this linear program in the following way:

$$\begin{aligned} \min_{y\in\mathbb{R}^m} \quad & b^T y \\ \text{s.t.} \quad & y \geqslant 0 \\ & A^T y \geqslant c \end{aligned} \tag{3.17}$$

The terms objective function, feasible solution, bounded, unbounded and optimal solution are defined analogously to the primal linear program.

It is clear that an equality $v^T x = d$ for some $v \in \mathbb{R}^n$ and some constant $d \in \mathbb{R}$ can be split into two inequalities $v^T x \leqslant d$ and $v^T x \geqslant d$. Similarly, an inequality $v^T x \geqslant d$ is equivalent to an inequality $(-v)^T x \leqslant -d$. Hence, we will also use equations and inequalities with both $\leqslant$ and $\geqslant$ when writing down a linear program. We can always transform such a linear program into the form (3.16).

**Remark 3.3.2.** The number of variables $x_i$ of the primal LP is equal to the number of inequalities $A^T y \geqslant c$ of the dual LP. Similarly, the number of variables $y_i$ of the dual LP is equal to the numbers of inequalities $Ax \leqslant b$ of the primal LP.

There is a beautiful notion of duality in the theory of linear programming. It says that the optima of the primal and the dual linear program are related. We have the following theorem.

**Theorem 3.3.3** (Weak duality). *If $x$ is feasible for the primal linear program (3.16) and $y$ is feasible for the dual linear program (3.17), then*

$$c^T x \leqslant b^T y.$$

*In particular, the optimum of the primal LP is bounded by the optimum of the dual LP and vice versa.*

For linear programs, there is an even stronger notion called *strong duality*.

**Theorem 3.3.4** (Strong duality). *If the primal LP (3.16) has an optimal solution $x^*$, then the dual LP (3.17) has an optimal solution $y^*$ such that*

$$c^T x^* = b^T y^*.$$

We now set up the linear programs for cliques and designs, first studied by Delsarte in [19, Section 3.2].

Let $(X, \mathcal{R})$ be a symmetric association scheme with $n$ classes and let $P$ and $Q$ be the matrices of (dual) eigenvalues. Since $(X, \mathcal{R})$ is symmetric, all adjacency matrices of the relations are symmetric and thus have real eigenvalues. Hence, $P$ and $Q = |X|^{-1}P^{-1}$ are real matrices.

Let $Y \subseteq X$ be a non-empty subset of $X$ and let $a$ be its inner distribution. Recall that by Lemma 3.2.2, we have $a \geqslant 0$ and $a_0 + \ldots + a_n = |Y|$. If $Y$ is a $D$-clique for some subset $D \subseteq \{1, \ldots, n\}$, then we additionally have $a_d = 0$ for every $d \in \{1, \ldots, n\} \backslash D$. Furthermore, we have that the dual distribution $a' = Qa$ of $Y$ is a linear transform of $a$ and non-negative by Lemma 3.2.7. Thus, writing $1^T = (1, \ldots, 1) \in \mathbb{R}^{n+1}$, we see that the inner distribution of a $D$-clique is a feasible solution to the following linear program:

$$\begin{aligned} \max_{x \in \mathbb{R}^{n+1}} \quad & 1^T x = x_0 + \ldots + x_n \\ \text{s.t.} \quad & x_0 = 1 \\ & x_i = 0 \text{ for } i \in \{1, \ldots, n\} \backslash D \\ & x_i \geqslant 0 \text{ for } i \in D \\ & Qx \geqslant 0 \end{aligned} \tag{3.18}$$

Notice that without the condition $Qx \geqslant 0$, the linear program (3.18) is unbounded.

We can now state Delsarte's linear programming bound (see [19]).

**Theorem 3.3.5.** *Let $(X, \mathcal{R})$ be a symmetric association scheme with $n$ classes and let $D \subseteq \{1, \ldots, n\}$. Then the linear program (3.18) has at least one feasible solution and is bounded. Moreover, if $LP(D)$ denotes the optimum of the linear program (3.18) and $Y \subseteq X$ is a $D$-clique with inner distribution $a = (a_0, \ldots, a_n)$, then $(a_0, \ldots, a_n)$ is a feasible solution of (3.18) and we have*

$$|Y| \leqslant LP(D).$$

We call this the *linear programming bound* for cliques and codes.

Notice that we can only set up the linear program (3.18) if $Q$ is real. This is the case for symmetric association schemes. If we wanted to apply the linear programming method to non-symmetric association schemes whose eigenvalues can be complex, we would have to define a different linear program involving real and imaginary parts. However, we do not need that in this thesis.

A nice application of the linear programming method is the *clique-coclique bound*. Its proof uses the linear programming bound (see [19, Theorem 3.9] for details).

**Theorem 3.3.6.** *Let $(X, \mathcal{R})$ be an association scheme, $D \subseteq \{1, \ldots, n\}$ and $\widetilde{D} = \{1, \ldots, n\} \setminus D$. If $Y \subseteq X$ is a $D$-clique and $Z \subseteq X$ is a $\widetilde{D}$-clique, then*

$$|Y| \cdot |Z| \leqslant |X|.$$

**Example 3.3.7.** The orthogonal array

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

from Example 3.2.12 has inner distribution $a = (1, 0, 3, 0)$. Hence, it is a $\{2\}$-clique in $H(3, 2)$. Since the base set of $H(3, 2)$ has 8 elements, it follows from Theorem 3.3.6 that a $\{1, 3\}$-clique in $H(3, 2)$ can have no more than 2 elements. In other words, a set of codewords in $\mathbb{F}_2^3$ such that any two distinct codewords have Hamming distance 1 or 3 can have no more than 2 elements. Any pair of binary words of Hamming distance 1 or 3 is an example of a $\{1, 3\}$-clique.

We can also set up a linear program for $T$-designs. Let $Y \subseteq X$ be a non-empty subset of $X$ and let $a'$ be its dual distribution. Recall that by Lemma 3.2.7, we have $a' \geqslant 0$ and $a'_0 + \ldots + a'_n = |X|$. If $Y$ is a $T$-design for some subset $T \subseteq \{1, \ldots, n\}$, then we additionally have $a'_t = 0$ for every $t \in T$. Furthermore, we have that the inner distribution $a = Q^{-1}a' = |X|^{-1}P\,a'$ of $Y$ is a linear transform of $a'$ and non-negative by Lemma 3.2.2. We also have $a'_0 = |Y|$. Thus, writing $1^T = (1, \ldots, 1) \in \mathbb{R}^{n+1}$, we see that the dual distribution $a'/|Y|$ of a $T$-design scaled by $|Y|$ is a feasible solution to the following linear program:

$$\begin{aligned} \max_{x \in \mathbb{R}^{n+1}} \quad & 1^T x = x_0 + \ldots + x_n \\ \text{s.t.} \quad & x_0 = 1 \\ & x_i = 0 \text{ for } i \in T \\ & x_i \geqslant 0 \text{ for } i \in \{1, \ldots, n\} \backslash T \\ & Px \geqslant 0 \end{aligned} \tag{3.19}$$

We can now state Delsarte's linear programming bound for designs.

**Theorem 3.3.8.** *Let $(X, \mathcal{R})$ be a symmetric association scheme with $n$ classes and let $T \subseteq \{1, \ldots, n\}$. Then the linear program (3.19) has at least one feasible solution and is bounded. Moreover, if $LP(T)$ denotes the optimum of the linear program (3.19) and $Y \subseteq X$ is a $T$-design with dual distribution $a' = (a'_0, \ldots, a'_n)$, then $\frac{1}{|Y|}(a'_0, \ldots, a'_n)$ is a feasible solution of (3.19) and we have*

$$\frac{|X|}{|Y|} \leqslant LP(T).$$

*Rearranging, we obtain the lower bound*

$$|Y| \geqslant \frac{|X|}{LP(T)}$$

*for the size of a $T$-design.*

We call this the *linear programming bound* for designs.

**Example 3.3.9.** Consider again the orthogonal array

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

from Example 3.2.12. We have $Y = \{000, 011, 101, 110\}$, so $|Y| = 4$ and the dual distribution of $Y$ is $a' = (4, 0, 0, 4)$. The scaled dual distribution is

$$\frac{a'}{|Y|} = (1, 0, 0, 1).$$

In order to set up the linear program for $T$-designs in $H(3, 2)$, we need the eigenvalues. The eigenvalues of the Hamming scheme are given in terms of

Krawtchouk polynomials (see [19, Theorem 4.2]). We find that the $P$-matrix of $H(3,2)$ is

$$P = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & -1 & -3 \\ 3 & -1 & -1 & 3 \\ 1 & -1 & 1 & -1 \end{pmatrix}.$$

Thus, we can set up the linear program (3.19) for $T = \{1,2\}$:

$$\begin{aligned} \max_{x \in \mathbb{R}^4} \quad & x_0 + x_1 + x_2 + x_3 \\ \text{s.t.} \quad & x_0 = 1 \\ & x_1 = 0 \\ & x_2 = 0 \\ & x_3 \geqslant 0 \\ & Px \geqslant 0 \end{aligned}$$

We see that $x_3$ is the only variable with an inequality constraint. Comparing the first and last columns of $P$, we see that the constraint $Px \geqslant 0$ implies $x_3 \leqslant x_0 = 1$. Hence, the linear program has a unique optimal solution $(1,0,0,1)$ with optimum 2. It follows from Theorem 3.3.8 that any $\{1,2\}$-design $D$ in $H(3,2)$ satisfies

$$|D| \geqslant \frac{8}{2} = 4.$$

This should not come as a surprise because by Theorem 3.2.13, a $\{1,2\}$-design in $H(3,2)$ is an orthogonal array of strength 2. Since any two columns of such an array must contain the 4 vectors $00, 01, 10, 11$, it is clear that a $\{1,2\}$-design must have at least 4 elements, agreeing with the linear programming bound. Furthermore, note that by strong duality (Theorem 3.3.4), the dual linear program also has an optimal solution and the optimum is the same.

In general, the linear programming method is a powerful tool to deduce results about the existence and non-existence of cliques and designs. For example, if the above linear program had an optimum strictly smaller than 2, then we could have deduced $|Y| > 4$. This would have implied that there are no non-trivial orthogonal arrays of strength 2 with 3 columns and entries 0 and 1 (that is the only orthogonal array would have been the trivial array containing all binary words of length 3).

This shows how linear programming can be used to show the non-existence of combinatorial structures. Furthermore, it shows that it is fruitful to rephrase combinatorial problems in the language of association schemes because this gives access to the linear programming method.

# 4 CODES AND DESIGNS IN THE SYMMETRIC GROUP

In this chapter, we give an overview of well-established results about subsets of the symmetric group with interesting combinatorial properties. They turn out to be cliques and designs in the association scheme of the symmetric group.

We first describe the association scheme of the symmetric group in more detail, specialising results from the previous chapters to the symmetric group. Then, we briefly show how error-correcting codes in the symmetric group can be understood in the context of association schemes. Finally, we investigate the connection between transitive subgroups and designs in the symmetric group. Many of our original contributions deal with generalisations and analogues of the results about designs in the symmetric group.

## 4.1 THE ASSOCIATION SCHEME OF THE SYMMETRIC GROUP

We start by examining the association scheme of the symmetric group in more detail. It is the conjugacy class scheme of the group $S_n$.

Recall that the conjugacy classes of the symmetric group correspond to the cycle types. The cycle type of a permutation $\sigma \in S_n$ is a partition $\lambda$ of $n$ and there exists a bijection between the conjugacy classes of the symmetric group and the partitions of $n$. Thus, the relations, the adjacency matrices, the eigenvalues, the inner distribution and the dual distribution are indexed by the partitions of $n$. The trivial character $1_{S_n}$ of the symmetric group is the irreducible character $\chi^{(n)}$ indexed by the partition $(n)$.

Notice that the identity $\mathrm{id} \in S_n$ is the only permutation of cycle type $(1^n)$. Hence, the partition $(1^n)$ indexes the conjugacy class $\{\mathrm{id}\}$ and the relation $R_{(1^n)}$ plays the role of the relation $R_0$.

Further, note that even though both the conjugacy classes and the irreducible characters are indexed by the partitions of $n$, there is a difference in which partition indexes the trivial conjugacy class and character. The partition $(1^n)$ indexes the trivial conjugacy class but the partition $(n)$ indexes the trivial character. This is an effect of the many duality properties in association schemes, see for example Table 3.1 and Table 3.2.

The eigenvalues of the conjugacy class scheme are given in Theorem 3.1.9. We find that

$$P_\mu(\lambda) = \frac{|C_\mu|}{\deg(\chi^\lambda)}\,\chi^\lambda(C_\mu) \quad \text{and} \quad Q_\lambda(\mu) = \deg(\chi^\lambda)\,\chi^\lambda(C_\mu)$$

for every $\lambda, \mu \vdash n$. We know from Theorem 2.2.7 that the irreducible characters of the symmetric group are integer-valued, hence we can ignore the complex conjugation in Theorem 3.1.9.

We specialise some results from Chapter 3 to this particular association scheme.

**Lemma 4.1.1.** *Let $\lambda, \mu \vdash n$ and let $P_\mu(\lambda)$ and $Q_\lambda(\mu)$ be the eigenvalues and dual eigenvalues of the conjugacy class scheme of the symmetric group. Then we have*

(*a*) $P_{(1^n)}(\lambda) = 1$,

(*b*) $Q_{(n)}(\mu) = 1$,

(*c*) $P_\mu((n)) = |C_\mu|$,

(*d*) $Q_\lambda((1^n)) = \deg(\chi^\lambda)^2$.

Using the dual eigenvalues, we can write down the minimal idempotents. We find that the idempotent $E_\lambda \in \mathbb{C}(S_n, S_n)$ is given by

$$E_\lambda(\sigma, \tau) = \frac{\deg(\chi^\lambda)}{n!}\,\chi^\lambda(\sigma^{-1}\tau).$$

In particular, since $\chi^{(n)} = 1_{S_n}$ is the trivial character, we find

$$E_{(n)}(\sigma, \tau) = \frac{1}{n!},$$

so that $E_{(n)}$ is the idempotent $\frac{1}{|X|}J$ that is in general called $E_0$.

For a non-empty subset $Y \subseteq S_n$, the dual distribution $a'$ of $Y$ is given by

$$a'_\lambda = \frac{\deg(\chi^\lambda)}{|Y|} \sum_{\sigma,\tau \in Y} \chi^\lambda(\sigma^{-1}\tau).$$

If the subset $Y$ is a subgroup of $S_n$, then the product $\sigma^{-1}\tau$ yields every element of $Y$ precisely $|Y|$ times as $\sigma$ and $\tau$ run through the elements of $Y$. Hence, the formula for the dual distribution simplifies to

$$a'_\lambda = \deg(\chi^\lambda) \sum_{\sigma \in Y} \chi^\lambda(\sigma). \tag{4.1}$$

Notice that

$$a'_{(n)} = \frac{1}{|Y|} \sum_{\sigma,\tau \in Y} 1 = |Y|,$$

again agreeing with the partition $(n)$ indexing the trivial character.

## 4.2 PERMUTATION CODES

In this section, we briefly look at so called *permutation codes*. Here, two permutations are 'close' to each other if they agree on many elements. See [7] for more information on permutation codes and [62] for a treatment of permutation codes using the theory of association schemes.

**Theorem 4.2.1.** *The map*

$$d : S_n \times S_n \to \{0, \dots, n\},\ d(\sigma, \tau) = |\{i \in \{1, \dots, n\} \mid \sigma(i) \neq \tau(i)\}|$$

*defines a metric on the symmetric group. It holds that*

$$d(\sigma, \tau) = n - |Fix(\sigma^{-1}\tau)|$$

*for every $\sigma, \tau \in S_n$.*

We call the metric $d$ the *fixed point metric*. Writing a permutation $\sigma$ as a tuple $(\sigma(1), \dots, \sigma(n))$, we see that the fixed point metric is equal to the Hamming distance on the set of tuples with $n$ entries from the set $\{1, \dots, n\}$ all of which are pairwise distinct. Since $d$ is a metric, we can define error-correcting codes in the usual way: A subset $C \subseteq S_n$ is called a *d-code* if any two distinct permutations in $C$ have at least distance $d$.

Note that we can read off the distance between two permutations $\sigma, \tau \in S_n$ from the cycle type $\lambda$ of $\sigma^{-1}\tau$. The fixed points of $\sigma^{-1}\tau$ correspond precisely to the 1s in the partition $\lambda$. The number of 1s in $\lambda$ is equal to $\lambda'_1 - \lambda'_2$, so we find

$$d(\sigma, \tau) = d \iff \lambda'_1 - \lambda'_2 = n - d.$$

In other words, two permutations have distance at least $d$ if and only if the cycle type of $\sigma^{-1}\tau$ has at most $n - d$ 1s. The distance between two permutations depends only on the cycle type of $\sigma^{-1}\tau$. Hence, we can understand permutation codes as $D$-cliques in the association scheme of the symmetric group.

**Theorem 4.2.2.** *Let $C$ be a non-empty subset of $S_n$ and let $d \in \{1, \dots, n\}$. Let $D \subseteq \mathrm{Par}_n$ be the set of all partitions of $n$ with at most $n - d$ 1s. Then*

$$C \text{ is a } d\text{-code} \iff C \text{ is a } D\text{-clique}.$$

This result allows using Delsarte's linear programming method (Theorem 3.3.5) to obtain bounds on the size of permutation codes.

A well-known upper bound for the size of permutation codes is the following.

**Theorem 4.2.3** ([7, Sec. II])**.** *Let $d \in \{1, \ldots, n\}$ and let $Y \subseteq S_n$ be a $d$-code. Then*

$$|Y| \leqslant \frac{n!}{(d-1)!} = n \cdot (n-1) \cdot \ldots \cdot d.$$

This bound was also obtained by Tarnanen [62] using the dual of Delsarte's linear programming method. In the paper, different linear programs are used together with character theory to derive further bounds on the size of $d$-codes. We see how powerful the combination of association schemes, linear programming and representation theory can be.

**Remark 4.2.4.** We emphasise that the metric $d$ does *not* give rise to an association scheme. That is, the relations

$$R_i = \{(\sigma, \tau) \in S_n \times S_n \mid d(\sigma, \tau) = i\}$$

for $i = 0, \ldots, n$ do *not* define an association scheme if $n \geqslant 4$. Notice that for $n \in \{1, 2, 3\}$, a partition of $n$ is uniquely determined by its number of 1s, so the relations $R_i$ are actually the relations of the conjugacy class scheme and so they do define an association scheme (ignoring the empty relation $R_{n-1}$).

In order to see that the relations $R_i$ do not define an association scheme for $n \geqslant 4$, notice that there are two types of permutations with precisely $n - 4$ fixed points: 4-cycles and products of 2 disjoint 2-cycles.

Take for example $\sigma = (1, 2, 3, 4) \in S_n$ and $\tau = (1, 2)(3, 4) \in S_n$. Then we have $d(\mathrm{id}, \sigma) = d(\mathrm{id}, \tau) = 4$, so $(\mathrm{id}, \sigma), (\mathrm{id}, \tau) \in R_4$ and the number $p^4_{22}$ of permutations having distance 2 from both the identity and $\sigma$ should be the same as the number of permutations having distance 2 from both the identity and $\tau$. The permutations having distance 2 from the identity are precisely the transpositions. Hence, we have to count how many of the permutations having distance 2 from $\sigma$ or $\tau$ are transpositions.

Notice that the permutations at distance 2 from $\sigma$ are precisely the permutations $(i, j) \circ \sigma$ for $i, j \in \{1, \ldots, n\}$ with $i \neq j$. We know that $(1, 2, 3, 4) = (1, 2)(2, 3)(3, 4)$ can be written as a product of 3 transpositions and is not a transposition itself. Hence, as $(1, 2, 3, 4)$ is an odd permutation, $(1, 2, 3, 4)$ cannot be written as the product of 2 transpositions. It follows that none of the permutations $(i, j) \circ \sigma$ are transpositions and thus there are no permutations at distance 2 from both the identity and $\sigma$.

On the other hand, it is easy to see that the transpositions $(1, 2)$ and $(3, 4)$ have distance 2 from $\tau$ (in fact these are the only transpositions with that property). It follows that there are 2 permutations at distance 2 from both the identity and $\tau$ and thus the relations $R_i$ do not define an association scheme.

We see that even though the fixed point metric does not define an association scheme, we can still investigate codes with respect to the metric by studying an association scheme whose relations are 'finer' than the ones given by the metric. They carry more information about pairs of permutations than just their distance. We have to pick the right set $D$ in order to study permutation codes using association schemes.

We will come back to permutation codes when studying the generalised symmetric group in Chapter 5.

While permutation codes are sets of permutations that are far apart from each other, one can study the opposite problem of investigating sets of permutations such that all of them are close to each other in the fixed point metric. More specifically, a subset $Y \subseteq S_n$ of permutations such that any two elements have *at most* distance $n - t$ is called *$t$-intersecting*. For any two permutations $\sigma, \tau$ of a $t$-intersecting set, there exist at least $t$ distinct elements $i \in \{1, \ldots, n\}$ such that $\sigma(i) = \tau(i)$. In the language of association schemes, $t$-intersecting sets are cocliques. They are studied in the area of Erdős-Ko-Rado combinatorics and there is a lot of literature dealing with the topic. We will not delve into this area further and refer the reader to [27] instead.

## 4.3 TRANSITIVE SUBSETS OF THE SYMMETRIC GROUP

In this section, we present results about transitive subsets of the symmetric group. We give a classic result by Livingstone and Wagner [42] and then present the results due to Martin and Sagan [46] who set the Livingstone–Wagner theorem in the context of association schemes and proved generalisations.

We start with a brief overview of transitive group actions (see [13, Chapter 1] for more details). Recall that an action of a group $G$ on a set $\Omega$ is called *transitive* if for every $x, y \in \Omega$ there exists $g \in G$ such that $g \cdot x = y$. If the element $g$ is unique for every choice of $x$ and $y$, then the action is called *sharply transitive*.

If a group $G$ acts transitively on a set $\Omega$, then all of its subgroups also act on $\Omega$ and we can ask which subgroups act transitively on $\Omega$. This can particularly be interesting when $G$ has a 'strong' sense of transitivity (like the symmetric group) and we are looking for subgroups with a 'weaker' sense of transitivity. Let us give some examples.

Recall that if $G$ acts on $\Omega$, then $G$ also acts on sets and tuples of elements of $\Omega$ in the natural way. If $t$ is an integer with $1 \leqslant t \leqslant |\Omega|$, we denote by $\Omega_t$ the set of all subsets of $\Omega$ with precisely $t$ elements. Then $G$ acts on $\Omega_t$ via

$$g \cdot \{x_1, \ldots, x_t\} = \{g \cdot x_1, \ldots, g \cdot x_t\}$$

for every $g \in G$ and $\{x_1, \ldots, x_t\} \in \Omega_t$. Similarly, $G$ acts on the tuples $\Omega^t$ via

$$g \cdot (x_1, \ldots, x_t) = (g \cdot x_1, \ldots, g \cdot x_t)$$

for every $g \in G$ and $(x_1, \ldots, x_t) \in \Omega^t$.

These actions give rise to two important notions of transitivity.

**Definition 4.3.1.** Let $G$ be a group acting on a set $\Omega$ and let $t$ be an integer with $1 \leqslant t \leqslant |\Omega|$. The group $G$ is called *t-transitive* if $G$ acts transitively on the set of all $t$-tuples in $\Omega^t$ whose entries are pairwise distinct. Furthermore, $G$ is called *t-homogeneous* if $G$ acts transitively on the set $\Omega_t$ of all subsets of $\Omega$ of size $t$.

It is clear that the symmetric group $S_n$ acts (sharply) $n$-transitive on the set $\{1, \ldots, n\}$.

**Example 4.3.2.** A *Latin square* of order $n$ is an $(n \times n)$ array filled with symbols from an alphabet $A$ of size $n$ such that in every row and every column every element of $A$ appears exactly once.

Consider the symmetric group $S_5$ on 5 elements. For a nice visualisation, let us represent the 5 elements by colours. Then we can write every permutation $\sigma \in S_5$ as a sequence of colours.

Figure 4.1 shows a Latin square of order 5. The row above the square fixes an ordering of the colours so that we get an identification between the set $\{1, 2, 3, 4, 5\}$ and the set $\{$blue, red, green, grey, yellow$\}$.

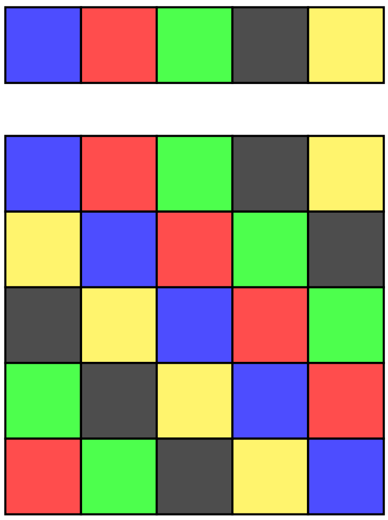


Figure 4.1: A transitive group, represented as a Latin square

The rows of the Latin square correspond to the elements of the cyclic subgroup $C = \langle(1, 2, 3, 4, 5)\rangle \subseteq S_5$. For example, the first line represents the identity. Clearly, $C$ acts transitively on the set $\{1, 2, 3, 4, 5\}$. We can also see that nicely from Figure 4.1: Every row contains each colour exactly once, so each row is a permutation $\sigma \in S_5$. Furthermore, every column contains every colour exactly once. Hence, we can map any colour to any colour using the 5 permutations.

It is clear that every $t$-transitive group also acts $t$-homogeneously. Furthermore, it is immediate that a group that acts $t$-transitively on a set $\Omega$ for $t \geqslant 2$ also acts $(t-1)$-transitively on $\Omega$. That this is also true for $t$-homogeneity is highly non-trivial and is a celebrated result by Livingstone and Wagner.

**Theorem 4.3.3** ([42, Theorem 2])**.** *Let $G$ be a subgroup of $S_n$ that is $t$-homogeneous for some $t$ with $1 \leqslant t \leqslant n/2$. Then $G$ is also $(t-1)$-homogeneous.*

This theorem was generalised by Martin and Sagan [46] in multiple ways. Their results include subsets instead of only subgroups and apply to a more general form of transitivity called $\lambda$-transitivity where $\lambda$ is a partition of $n$. Let us explain these notions.

It follows from the group axioms that if a finite group $G$ acts transitively on $\Omega$, then for any $x, y \in \Omega$ the number of $g \in G$ such that $g \cdot x = y$ is a constant independent of $x$ and $y$. This constant is the size of the stabiliser $\operatorname{Stab}(x)$. Since all stabilisers under a transitive group action are conjugate, all of them have the same size.

This is not necessarily the case any more if we want to investigate transitive *sets*. By taking a transitive subgroup and adding some other elements of the group to it, we obtain a subset of group elements that can map any element $x \in \Omega$ to any element $y \in \Omega$ but the number of group elements that do this may depend on the choice of $x$ and $y$. Such subsets are not Delsarte designs in the association scheme of the symmetric group. Hence, we make the following definition.

**Definition 4.3.4.** Let $G$ be a group acting on a set $\Omega$. A subset $Y \subseteq G$ is called *transitive* on $\Omega$ if there exists a constant $c > 0$ such that for any $a, b \in \Omega$, there are exactly $c$ elements $g \in Y$ with $g \cdot a = b$.

Note that $Y$ is sharply transitive if the constant $c$ in this definition is equal to 1. The definitions of $t$-transitivity and $t$-homogeneity carry over to subsets in the same way.

For every $x \in \Omega$, the integer $c$ is precisely the number of elements in $Y$ that stabilise $x$, so $c = |Y \cap \operatorname{Stab}(x)|$ is independent of $x$. If $Y$ is a subgroup, then this definition coincides with the definition of a transitive group action.

**Example 4.3.5.** Let us showcase the difference between transitive subgroups and transitive subsets.

First, consider the subgroup $Y = \langle(1,2,3,4,5),(1,2,4,3)\rangle \subseteq S_5$. It can be shown that $Y$ is a semidirect product $C_5 \rtimes C_4$ of cyclic groups and is sharply 2-transitive on the set $\{1,2,3,4,5\}$. In particular, $Y$ is also 2-homogeneous but clearly not *sharply* 2-homogeneous because there are elements $y, y' \in Y$ with $y \cdot (1,2) = (1,2)$ and $y' \cdot (1,2) = (2,1)$. But then we have

$$y \cdot \{1,2\} = y' \cdot \{1,2\} = \{1,2\}$$

for distinct elements $y$ and $y'$. It can easily be shown via an exhaustive computer search that the symmetric group $S_5$ does not have a sharply 2-homogeneous subgroup.

However, there exists a sharply 2-homogeneous subset of $S_5$. We present it using colours like in Example 4.3.2.

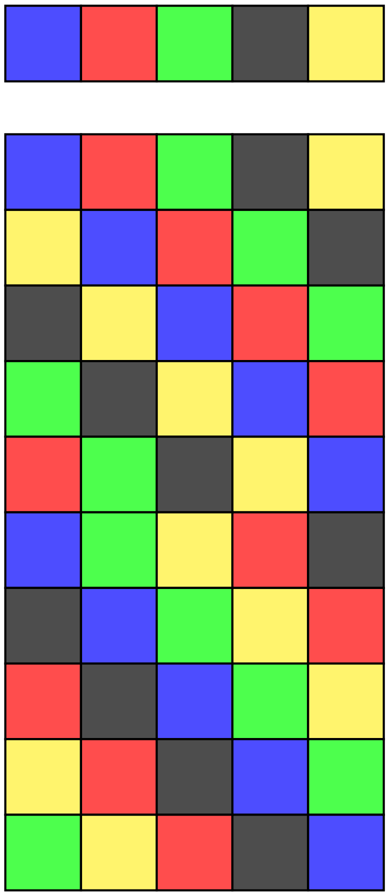


Figure 4.2: A 2-homogeneous subset of $S_5$

All 10 rows of this array are distinct and contain every colour exactly once. Hence, this is a set of 10 permutations in the symmetric group $S_5$. By taking enough time and checking carefully, one can verify that any two columns contain every unordered pair of two distinct colours exactly once. Thus, this set of permutations is sharply 2-homogeneous.

By taking a closer look, one can spot that this example is made up of two Latin squares stacked on top of each other. The first 5 rows represent the group $\langle(1,2,3,4,5)\rangle$ and the last 5 rows represent the coset $(1,2,4,3)\circ\langle(1,2,3,4,5)\rangle$. Their union is not a subgroup. The group generated by these 10 permutations is the 2-transitive group $Y$ from earlier which has order 20. If we write

$$D = \langle(1,2,3,4,5)\rangle \cup (1,2,4,3)\circ\langle(1,2,3,4,5)\rangle,$$

we find that both $D$ and $Y \setminus D$ are sharply 2-homogeneous subsets and their union $Y$ is a sharply 2-transitive group.

Observe that $D$ is also 1-transitive. The columns in Figure 4.2 contain every colour exactly twice. We see that this is in spirit with the Livingstone–Wagner theorem: A 2-homogeneous group is also 1-homogeneous (which is the same as 1-transitive). However, since the original Livingstone–Wagner theorem only applies to subgroups, we cannot apply it to our subset $D$. That the Livingstone–Wagner theorem also holds for subsets is one part of the generalisation of the Livingstone–Wagner theorem by Martin and Sagan.

One reason to study transitive sets that are not subgroups is their size. We have just seen that the smallest 2-homogeneous subgroup of $S_5$ has 20 elements while the smallest 2-homogeneous subset of $S_5$ has 10 elements. More generally, it is well-known (see for example [13, Theorem 4.11]) that for $t \geqslant 6$, the only $t$-transitive groups are $S_n$ (for $n \geqslant 6$) and $A_n$ (for $n \geqslant 8$). However, it is possible to construct $t$-transitive subsets of $S_n$ with $t \geqslant 6$ that are considerably smaller than $S_n$ or $A_n$. See [46, Section 6] for details.

The results by Martin and Sagan also deal with ordered set partitions of $\Omega$ and a corresponding notion of transitivity.

**Definition 4.3.6.** A subset $Y \subseteq S_n$ is called *$\lambda$-transitive* if it is transitive on the set of all $\lambda$-tabloids (ordered set partitions of shape $\lambda$, see Section 2.2).

These notions of transitivity include $t$-transitivity and $t$-homogeneity that we defined earlier.

If $G$ acts on the set $\{1,\dots,n\}$ and $S \subseteq \{1,\dots,n\}$ is any subset, then

$$g \cdot (\{1,\dots,n\} \setminus S) = \{1,\dots,n\} \setminus (g \cdot S)$$

for every $g \in G$. It follows that if $G$ acts on subsets $S$ of size $t$, it also acts on the pairs $(\{1,\dots,n\} \setminus S, S)$. If $1 \leqslant |S| = t \leqslant \frac{n}{2}$, then these pairs are precisely the $(n-t,t)$-tabloids. Hence, $G$ is $t$-homogeneous if and only if it acts transitively on the set of all $(n-t,t)$-tabloids. In other words, $(n-t,t)$-transitivity is the same as $t$-homogeneity.

Similarly, we can think of a tuple $(x_1,\dots,x_t) \in \{1,\dots,n\}^t$ with pairwise distinct entries as an ordered set partition with $t$ sets $\{x_1\},\dots,\{x_t\}$ of size 1 and one remaining subset of size $n-t$. Hence, $t$-transitivity is the same as $(n-t, 1, \ldots, 1)$-transitivity.

One of the main theorems in [46] is the following which is a generalisation of the Livingstone–Wagner theorem.

**Theorem 4.3.7** ([46, Theorem 3])**.** *Let $\lambda$ and $\mu$ be partitions of $n$ and let $Y$ be a $\lambda$-transitive subset of $S_n$. If $\mu \trianglerighteq \lambda$, then $Y$ is also $\mu$-transitive.*

Since $t$-homogeneity is the same as $\lambda$-transitivity for the partition $\lambda = (n-t, t)$, Theorem 4.3.7 can be applied to $t$-homogeneity. Taking $t \in \mathbb{N}$ with $1 \leqslant t \leqslant n/2$, $\lambda = (n-t, t)$ and $\mu = (n-t+1, t-1)$, we find that $\mu \trianglerighteq \lambda$. Thus, $t$-homogeneity implies $(t-1)$-homogeneity and Theorem 4.3.3 is an immediate corollary of Theorem 4.3.7. The proof is illustrated in Figure 4.3.

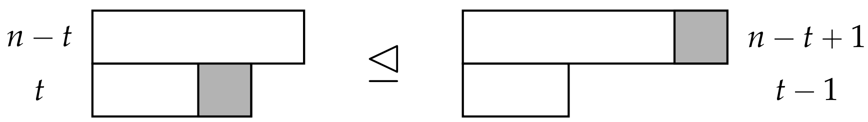



Figure 4.3: Proof of the Livingstone–Wagner theorem

Theorem 4.3.7 can be proved using the theory of association schemes. The key idea is to prove that a $\lambda$-transitive set $Y \subseteq S_n$ is a $T_\lambda$-design in the conjugacy class scheme of the symmetric group for some subset $T_\lambda \subseteq \mathrm{Par}_n$. Recalling that a $T$-design is also a $T'$-design for every subset $T' \subseteq T$, proving Theorem 4.3.7 comes down to comparing the subsets $T_\lambda$ for different partitions $\lambda$.

The following theorem gives a characterisation of $\lambda$-transitive subsets as Delsarte designs in the association scheme of the symmetric group.

**Theorem 4.3.8** ([46, Theorem 4])**.** *Let $Y \subseteq S_n$ be a non-empty subset with dual distribution $a'$ and $\lambda \vdash n$. Then*

$$Y \text{ is } \lambda\text{-transitive} \iff a'_\mu = 0 \text{ for every partition } \lambda \trianglelefteq \mu \neq (n).$$

Recall that the partition $(n)$ indexes the trivial character and that we have $a'_{(n)} = |Y|$ so we have to exclude the partition $(n)$ in the theorem.

The reader should interpret Theorem 4.3.8 as giving a bijection between partitions $\lambda$ (that is 'types' of transitivity in the context of $\lambda$-transitivity) and sets $T_\lambda \subseteq \mathrm{Par}_n$ of partitions of $n$ (that is 'types' of Delsarte designs in an association scheme). In other words, a $\lambda$-transitive subset $Y \subseteq S_n$ is a Delsarte $T_\lambda$-design where $T_\lambda = \{\mu \vdash n \mid \mu \trianglerighteq \lambda, \mu \neq (n)\}$.

Theorem 4.3.7 now follows easily from Theorem 4.3.8 because the dominance order is transitive. If we have partitions $\lambda, \mu \vdash n$ with $\mu \trianglerighteq \lambda$, then every partition that dominates $\mu$ also dominates $\lambda$. Hence, if the dual distribution $a'$

is equal to 0 at every partition that dominates $\lambda$, then in particular it is equal to 0 at every partition that dominates $\mu$. We obtain $T_\mu \subseteq T_\lambda$ so a $T_\lambda$-design is also a $T_\mu$-design. It follows that if $\mu \trianglerighteq \lambda$, then every $\lambda$-transitive set is also $\mu$-transitive.

We will derive similar characterisations of Delsarte designs for the wreath product $G \wr S_n$ where $G$ is a finite abelian group. Letting $G = C_1$ in our results (Theorem 5.3.11 and Theorem 5.5.11), we recover Theorems 4.3.7 and 4.3.8.

# 5 CODES AND DESIGNS IN THE GENERALISED SYMMETRIC GROUP

In [46], Martin and Sagan asked whether the results we described in Chapter 4 (Theorem 4.3.7 and Theorem 4.3.8) can be generalised to the Coxeter group of type $B_n$, the group of *signed* permutations. We answer this question affirmatively by considering wreath products of the form $G \wr S_n$ where $G$ is a finite abelian group. This includes the Coxeter group of type $B_n$ in the case $G = C_2$ and also the original results by Martin and Sagan for the symmetric group in the case $G = C_1$. We will see how designs in the conjugacy class scheme of the group $G \wr S_n$ can be interpreted using transitivity on *coloured partitions*, generalising the notion of $\lambda$-transitivity.

First, we recall some facts about transitive group actions and present the technique we use to derive characterisations of transitive sets as Delsarte designs. Then, we give a short introduction to wreath products and their character theory. We define a notion of transitivity and use our method to show that these transitivity types correspond to designs in the conjugacy class association scheme. We use our characterisation to give analogues of the Livingstone–Wagner theorem and give explicit constructions. Moreover, we look at special cases of designs and codes and use orthogonal polynomials to establish relations between them. Finally, we show how our results have applications to the existence question of finite projective planes, providing a new approach of proving the non-existence of a finite projective plane of order $p^n \pm 1$.

## 5.1 THE GENERAL METHOD

The aim of this chapter is to study the group action of the generalised symmetric group on certain sets $\Omega$. In this section, we first collect some basic results about transitive group actions. They are easily found in the literature, see for example [13] for details. Then, we describe our technique to study transitivity using permutation characters.

**Definition 5.1.1.** Let $G$ be a group acting on a set $\Omega$ and on another set $\Omega'$. The actions on $\Omega$ and $\Omega'$ are called *equivalent* if there exists a bijection $\varphi : \Omega \to \Omega'$ such that

$$\varphi(g \cdot x) = g \cdot \varphi(x)$$

for every $g \in G$ and $x \in \Omega$.

We can think of equivalent actions as being 'essentially the same'. An important result is that transitive actions are equivalent to actions on cosets (see for example [13, Theorem 1.3]).

**Lemma 5.1.2.** *Let $G$ be a group acting transitively on a set $\Omega$.*

(*a*) *For every $x, y \in \Omega$ we have*

$$\mathrm{Stab}(x) = g\,\mathrm{Stab}(y)g^{-1}$$

*for some $g \in G$.*

(*b*) *The action of $G$ on $\Omega$ is equivalent to the action of $G$ by left multiplication on the cosets $G/H = \{gH \mid g \in G\}$ where $H = \mathrm{Stab}(x)$ for some $x \in \Omega$.*

(*c*) *Two coset actions on $G/H$ and $G/K$ by left multiplication are isomorphic if and only if $H$ and $K$ are conjugate in the full symmetric group on $\Omega$.*

This lemma allows us to investigate transitivity solely within the group $G$ itself. Every subgroup $H$ of $G$ gives rise to a transitive action of $G$ (the action of $G$ on the cosets $G/H$) and every transitive action arises this way. We can think of each subgroup $H$ as a certain 'type' of transitivity.

In order to illustrate this point of view, let us look at $t$-transitivity in the symmetric group. The action of $S_n$ on $\{1, \ldots, n\}$ is $t$-transitive for every $t \leqslant n$. The stabiliser of a $t$-tuple $(x_1, \ldots, x_t)$ with pairwise distinct entries $x_i \in \{1, \ldots, n\}$ is

$$S = S_{\{1,\ldots,n\}\setminus\{x_1,\ldots,x_t\}} \cong S_{n-t} \cong S_1 \times \ldots \times S_1 \times S_{n-t}.$$

Notice that $S$ is a Young subgroup of shape $(n - t, 1, \ldots, 1)$. Thus, in view of Lemma 5.1.2, a subset $Y$ of $S_n$ is $t$-transitive on $\{1, \ldots, n\}$ if and only if it is transitive on the cosets $S_n/S$ where $S = S_{n-t}$. The action of $S_n$ on the cosets of $S_{n-t}$ is equivalent to the action of $S_n$ on $(n - t, 1, \ldots, 1)$-tabloids.

We now describe the method we use to connect transitive subsets to designs in an association scheme. The technique can be applied to any finite group $G$ acting on a set $\Omega$. The result is essentially that if the decomposition of the permutation character on $\Omega$ is known, then this decomposition gives rise to a characterisation of transitivity on $\Omega$ as a $T$-design in an association scheme.

Let $G$ be a finite group acting on a set $\Omega$. We define the *incidence matrix* of this action to be the matrix $M \in \mathbb{C}(G, \Omega \times \Omega)$ given by

$$M(g,(a,b)) = \begin{cases} 1, & \text{if } g \cdot a = b, \\ 0, & \text{otherwise.} \end{cases}$$

Denote by $\operatorname{colsp}(M)$ the vector space generated by the columns of $M$. The following theorem is crucial.

**Theorem 5.1.3.** *Let $G$ be a finite group acting on a set $\Omega$ with incidence matrix $M$. Let $\chi_0, \chi_1, \dots, \chi_n$ be the (complex) irreducible characters of $G$ and let $V_0, V_1, \dots, V_n$ be the corresponding eigenspaces of the conjugacy class association scheme of $G$. Define $I \subseteq \{0, 1, \dots, n\}$ such that $k \in I$ if and only if $\chi_k$ is an irreducible constituent of the permutation character $\xi$ of $G$ on $\Omega$. Then*

$$\operatorname{colsp}(M) = \bigoplus_{k \in I} V_k.$$

*Proof.* We have $MM^T \in \mathbb{C}(G,G)$ and for $g, h \in G$ we have

$$(MM^T)(g,h) = \sum_{(a,b) \in \Omega \times \Omega} M(g,(a,b))M(h,(a,b)).$$

The summands are either 0 or 1. A summand is equal to 1 if and only if $g \cdot a = b$ and $h \cdot a = b$, so $(g^{-1}h) \cdot a = a$. Thus, the entry of $MM^T$ at position $(g,h)$ is equal to the number of fixed points $|\operatorname{Fix}(g^{-1}h)|$ of $g^{-1}h$.

For the permutation character $\xi$, we have $\xi(g) = |\operatorname{Fix}(g)|$. Thus, considering the matrix $P \in \mathbb{C}(G,G)$ corresponding to $\xi$ given by

$$P(g,h) = \xi(g^{-1}h) = |\operatorname{Fix}(g^{-1}h)|,$$

we obtain $P = MM^T$. Recall that $\operatorname{colsp}(AA^T) = \operatorname{colsp}(A)$ holds for every real matrix $A$. This implies

$$\operatorname{colsp}(M) = \operatorname{colsp}(MM^T) = \operatorname{colsp}(P).$$

Now consider the decomposition

$$\xi = m_0\chi_0 + \ldots + m_n\chi_n$$

of $\xi$ into irreducible characters $\chi_k$ where $m_k \in \mathbb{N}_0$. Recall that by (3.6) and Theorem 3.1.9, the idempotent matrix $E_k$ of the conjugacy class association scheme is given by

$$E_k(g,h) = c_k\chi_k(g^{-1}h)$$

for some constant $c_k > 0$. Since

$$P(g,h) = \xi(g^{-1}h) = \sum_{k=0}^{n} m_k \chi_k(g^{-1}h) = \left(\sum_{k=0}^{n} m_k c_k^{-1} E_k\right)(g,h),$$

we obtain (since $k \in I$ if and only if $m_k \neq 0$)

$$P = \sum_{k=0}^{n} m_k c_k^{-1} E_k = \sum_{k\in I} m_k c_k^{-1} E_k. \tag{5.1}$$

Recall that $\operatorname{colsp}(E_k) = V_k$ and that the spaces $V_k$ are pairwise orthogonal. Hence, the column space of $P$ is contained in $\bigoplus_{k\in I} V_k$.

Conversely, let $k \in I$ and $v$ be a column vector of $E_k$. From (3.2), we have $E_k v = v$ and $E_l v = 0$ for $l \neq k$, so it follows from (5.1) that

$$Pv = PE_k v = m_k c_k^{-1} v.$$

Since $m_k \neq 0$ for $k \in I$, we conclude that $V_k = \operatorname{colsp}(E_k) \subseteq \operatorname{colsp}(P)$ for every $k \in I$. Thus, we have

$$\operatorname{colsp}(M) = \operatorname{colsp}(P) = \bigoplus_{k\in I} V_k. \qquad \square$$

Similar ideas of using incidence matrices have appeared in publications before (for example in [20, Section 3] under the name *Riemann matrix* or in the treatment of the symmetric group in [46, Section 4]). However, we could not find a reference for our approach using $G$ for the rows and $\Omega \times \Omega$ for the columns in the context of association schemes. Our approach applies to any transitive group action and we believe that it is a useful general method, especially for groups with a well-understood representation theory.

Now we can describe the connection between transitivity and the dual distribution.

**Theorem 5.1.4.** *Let $G$ be a finite group acting transitively on a set $\Omega$. Let $\chi_0, \chi_1, \dots, \chi_n$ be the (complex) irreducible characters of $G$ and define $I \subseteq \{0, 1, \dots, n\}$ such that $k \in I$ if and only if $\chi_k$ is an irreducible constituent of the permutation character $\xi$ of $G$ on $\Omega$. Let $Y$ be a non-empty subset of $G$ with dual distribution $(a'_0, a'_1, \dots, a'_n)$. Then $Y$ is transitive on $\Omega$ if and only if*

$$a'_k = 0 \quad \text{for each } k \in I \setminus \{0\}.$$

*Proof.* Note that $G$ acts transitively on $\Omega$ if and only if

$$M^T \mathbb{1}_G = c \cdot \mathbb{1}_{\Omega\times\Omega}$$

for some constant $c > 0$, since this equation means that for any pair $(a, b) \in \Omega \times \Omega$ there exist precisely $c$ elements of $G$ that map $a$ to $b$. Thus, $Y$ is transitive on $\Omega$ if and only if

$$M^T \mathbb{1}_Y = c' \cdot \mathbb{1}_{\Omega\times\Omega}$$

for some constant $c' > 0$. This constant can be calculated by observing that if $Y$ is transitive, then so are the cosets $gY$ for $g \in G$. Since the disjoint union of transitive sets is a transitive set, we find that $c = |G/Y| \cdot c'$ which gives

$$c' = \frac{|Y|}{|G|} \cdot c.$$

Now we see that $Y$ is transitive on $\Omega$ if and only if

$$\frac{1}{|Y|} M^T \mathbb{1}_Y = \frac{1}{|G|} M^T \mathbb{1}_G,$$

hence if and only if

$$\mathbb{1}_Y - \frac{|Y|}{|G|} \mathbb{1}_G$$

is orthogonal to the column space of $M$. By Theorem 5.1.3, the column space of $M$ is given by $\operatorname{colsp}(M) = \bigoplus_{k\in I} V_k$. Hence, $\mathbb{1}_Y - \frac{|Y|}{|G|}\mathbb{1}_G$ is orthogonal to the column space of $M$ if and only if it is orthogonal to the spaces $V_k$ with $k \in I$.

First, note that $V_0$ is spanned by $\mathbb{1}_G$ and that

$$\langle \mathbb{1}_Y - \frac{|Y|}{|G|} \mathbb{1}_G, \mathbb{1}_G \rangle = 0.$$

Now let $k \in \{1, \ldots, n\}$ and $v \in V_k$. Since the spaces $V_k$ are pairwise orthogonal and $\mathbb{1}_G \in V_0$, we find that

$$\langle \mathbb{1}_Y - \frac{|Y|}{|G|} \mathbb{1}_G, v \rangle = \langle \mathbb{1}_Y, v \rangle.$$

Thus, for $k \neq 0$, $\mathbb{1}_Y - \frac{|Y|}{|G|}\mathbb{1}_G$ is orthogonal to $V_k$ if and only if $\mathbb{1}_Y$ is orthogonal to $V_k$. Recall that by (3.10), $\mathbb{1}_Y$ is orthogonal to $V_k$ if and only if $a'_k = 0$, which concludes the proof. □

Let us illustrate how to use Theorem 5.1.4 by giving an alternative proof of Theorem 4.3.8 about $\lambda$-transitivity in the symmetric group.

**Example 5.1.5.** Consider the symmetric group $S_n$ and a partition $\lambda \vdash n$. Consider the action of $S_n$ on the set of $\lambda$-tabloids and let $\xi^\lambda$ be its permutation character. It follows from Young's rule (2.2) that

$$\langle \xi^\lambda, \chi^\mu \rangle \neq 0 \iff \mu \trianglerighteq \lambda.$$

Hence, we find that the set $I$ of partitions of $n$ indexing irreducible constituents of the permutation character $\xi^\lambda$ is precisely the set

$$I = \{\mu \vdash n \mid \mu \trianglerighteq \lambda\}.$$

As the partition $(n)$ indexes the trivial character and hence corresponds to character $\chi_0$ in Theorem 5.1.4, Theorem 4.3.8 now follows immediately from Theorem 5.1.4.

We see that in view of Theorem 5.1.4, proving a characterisation of transitive sets as Delsarte designs in the conjugacy class scheme is essentially equivalent to decomposing the permutation character of the action. Hence, our next task is to study the representation theory of $G \wr S_n$ and decompose the permutation characters $\xi_H$ for a given subgroup $H$.

## 5.2 WREATH PRODUCTS

We now turn to wreath products. The generalised symmetric group that we are interested in is a wreath product of an abelian group with the symmetric group.

From now on, we call the group that we investigate $W$. It stands for both *wreath product* and *Weyl group* (some special cases of the generalised symmetric group are in fact Weyl groups). For the rest of this chapter, $(G, +)$ is a finite abelian group.

The wreath product $W = G \wr S_n$ is a group, namely the semidirect product $W = G^n \rtimes S_n$ where $\sigma \in S_n$ acts on $G^n$ from the left by mapping $(g_1, \dots, g_n) \in G^n$ to $\sigma g = (g_{\sigma^{-1}(1)}, \dots, g_{\sigma^{-1}(n)})$. The group $G^n$ is called the *base group* of the wreath product $G \wr S_n$. We can represent every element of $W$ as a pair $(g, \pi)$ with $g \in G^n$ and $\pi \in S_n$. The product of two elements $(f, \sigma), (g, \pi) \in W$ is given by

$$(f, \sigma)(g, \pi) = (f + \sigma g, \sigma\pi).$$

For more information about wreath products, see [23, Chapter 2.6] for example.

If $G = C_1$ or $G = C_2$, then $W$ is the Coxeter group of type $A_{n-1}$ or $B_n$, respectively. If $G = C_r$ is the cyclic group of order $r$, then $W$ is the complex reflection group $G(r, 1, n)$. The group $C_r \wr S_n$ is also called the *generalised symmetric group*. For more information about Coxeter groups and complex reflection groups, we refer the reader to [32].

Given an integer $n \in \mathbb{N}$, we write $[n]$ for the set $\{1, \dots, n\}$. There is an action of $W$ on the set $M = G \times [n]$. For $((g_1, \dots, g_n), \pi) \in W$ and $(c, i) \in G \times [n]$, it is given by

$$((g_1, \dots, g_n), \pi) \cdot (c, i) = (c + g_{\pi(i)}, \pi(i)).$$

It is easy to see that the wreath product $W$ consists of all permutations $\sigma$ of the set $M$ such that

$$\sigma(0, i) = (c, j) \implies \sigma(g, i) = (g + c, j)$$

for all $g \in G$ and all $i \in [n]$ together with composition of mappings. We can think of the set $M$ as the set of $|G|$ copies of $[n]$ where each $i \in [n]$ appears in $|G|$ different colours.

**Remark 5.2.1.** We can give a nice visualisation of the action of the group $C_r \wr S_n$ using coloured arrays of boxes. Consider for example the group $C_3 \wr S_4 = C_3^4 \rtimes S_4$. It acts on the coloured numbers 1, 2, 3 and 4 where every number exists in 3 possible colours. If we choose the colours to be blue, red and green, then the set $M = C_3 \times [4]$ that $C_3 \wr S_4$ acts on are the 12 cells of the following coloured rectangle.

| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| 1 | 2 | 3 | 4 |
| 1 | 2 | 3 | 4 |

The symmetric group $S_4$ permutes the columns as a whole (without changing the order of the boxes in the columns) while the base group $C_3^4$ permutes the boxes in the columns cyclically and independently. These two basic actions are visualised in Figure 5.1.

| | | | |
|---|---|---|---|
| 2 | 1 | 4 | 3 |
| 2 | 1 | 4 | 3 |
| 2 | 1 | 4 | 3 |

sample action of $S_4$

| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| 1 | 2 | 3 | 4 |
| 1 | 2 | 3 | 4 |

sample action of $C_3^4$

Figure 5.1: The action of the generalised symmetric group

The generalised symmetric group $C_r \wr S_n$ consists of all permutations that can be obtained as compositions of the actions of $S_n$ and $C_r^n$. In general, a generalised permutation $((g_1, \ldots, g_n), \pi) \in C_r \wr S_n$ acts on the coloured rectangle by first permuting the columns according to the permutation $\pi$ and then cyclically shifting the colours in the columns according to the elements $g_1, \ldots, g_n$.

### 5.2.1 *Conjugacy Classes*

We now describe the conjugacy classes of $G \wr S_n = G^n \rtimes S_n$, where $G$ is a finite abelian group. For further reading, we refer the reader to [45, Appendix B, Chapter I]. We are interested in the following theorem characterising the conjugacy classes.

**Theorem 5.2.2.** *Let $G$ be a finite group. The conjugacy classes of $G \wr S_n$ are indexed by functions $\underline{\lambda} : G \to \mathrm{Par}$ such that*

$$\sum_{g \in G} |\underline{\lambda}(g)| = n.$$

We write

$$\Lambda_n(G) = \{\underline{\lambda} : G \to \mathrm{Par} \mid \sum_{g \in G} |\underline{\lambda}(g)| = n\}$$

for the set of all these partition valued functions $\underline{\lambda}$. It can be helpful to think of $\underline{\lambda}$ as a tuple of partitions with entries indexed by the elements of $G$.

The indexing of the conjugacy classes arises as follows. Consider an element $((g_1, \dots, g_n), \pi) \in G \wr S_n$. The permutation $\pi \in S_n$ can be written as a product of disjoint cycles of the form $(i_1, \dots, i_k)$. It turns out that the element $g_{i_1} + \dots + g_{i_k}$ is determined up to conjugacy. Thus, since $G$ is abelian in our setting, it is determined uniquely. We call this element $g_{i_1} + \dots + g_{i_k}$ the *sign* of the cycle $(i_1, \dots, i_k)$.

For each $g \in G$, we can determine all the cycles in $\pi$ that have sign $g$. Writing down all the cycle lengths that occur and ordering them decreasingly, we obtain a partition associated with the element $g$. The *cycle type* of an element $((g_1, \dots, g_n), \pi) \in G \wr S_n$ is defined to be the mapping $\underline{\lambda} : G \to \mathrm{Par}$ where $\underline{\lambda}(g)$ equals the partition of cycle lengths of $\pi$ that have sign $g$. Then two elements of $G \wr S_n$ are conjugate if and only if they have the same cycle type. Since the sum of all cycle lengths of $\pi$ is $n$, the conjugacy classes of $G \wr S_n$ are indexed by the set $\Lambda_n(G)$.

We denote by $C_{\underline{\lambda}}$ the conjugacy class indexed by $\underline{\lambda}$. In the case $G = C_1$, $\underline{\lambda}$ is a tuple with one entry, so $\underline{\lambda}$ is just a partition, agreeing with $C_1 \wr S_n = S_n$ being the symmetric group.

**Example 5.2.3.** The conjugacy class of the identity of $G \wr S_n$ is indexed by the element $\underline{\lambda} \in \Lambda_n(G)$ with $\underline{\lambda}(0) = (1^n)$ and $\underline{\lambda}(g) = \varnothing$ for every $g \neq 0$. More generally, consider an element of $C_r^n$, that is an element of the form $((g_1, \dots, g_n), \mathrm{id})$. Its conjugacy class is indexed by the element $\underline{\lambda} \in \Lambda_n(G)$ with $\underline{\lambda}(g) = (1^{m_g})$ where $m_g$ is the number of times that $g$ appears in the tuple $(g_1, \dots, g_n)$.

For another example, consider $C_2 \wr S_5$, where we write $C_2 = \{1, -1\}$ multiplicatively, acting on the set $\{1, -1, 2, -2, 3, -3, 4, -4, 5, -5\}$. Then

$$((1, -1, -1, -1, 1), (1, 2, 3)(4, 5))$$

is a generalised permutation. The cycle type of $(1, 2, 3)(4, 5)$ in $S_5$ is the partition $(3, 2)$. We compute the signs

$$1 \cdot (-1) \cdot (-1) = 1 \text{ and } (-1) \cdot 1 = -1.$$

Hence, the conjugacy class of $((1, -1, -1, -1, 1), (1, 2, 3)(4, 5))$ is indexed by the element $\underline{\lambda} \in \Lambda_5(C_2)$ with $\underline{\lambda}(1) = (3)$ and $\underline{\lambda}(-1) = (2)$. We can also write this as a pair of partitions $\underline{\lambda} = ((3), (2))$.

We can also write down the cycle decomposition of our generalised permutation inside the symmetric group on the set $\{1, -1, 2, -2, 3, -3, 4, -4, 5, -5\}$ and find

$$(1, -2, 3)(-1, 2, -3)(4, 5, -4, -5).$$

In general, if we consider the case $G = C_2 = \{1, -1\}$, the cycle decomposition consists of pairs of cycles $(x_1, \ldots, x_k)(-x_1, \ldots, -x_k)$ of length $k$ and cycles of the form $(x_1, \ldots, x_k, -x_1, \ldots, -x_k)$ of length $2k$. For cycles of the first type, we put their length into the partition $\underline{\lambda}(1)$, while for cycles of the second type, we put half their length into the partition $\underline{\lambda}(-1)$. This gives an alternative interpretation of the conjugacy classes for the group $C_2 \wr S_n$ of signed permutations.

### 5.2.2 *Characters of Wreath Products*

Now, we review the representation theory of wreath products of the form $G \wr S_n = G^n \rtimes S_n$, where $G$ is a finite abelian group. We first consider the more general case $A \rtimes H$, where $A$ and $H$ are finite groups with $A$ being abelian. We essentially follow [57, Sect. 8.2].

Since $A$ is abelian, each irreducible character of $A$ has degree 1 and the irreducible characters $\theta$ of $A$ form a group $A'$. Note that $A$ is normal in $A \rtimes H$, so the group $H$ acts on $A'$ by

$$(h\theta)(a) = \theta(h^{-1}ah) \quad \text{for } h \in H, \theta \in A' \text{ and } a \in A.$$

Let $\theta$ be an irreducible character of $A$ and let $K$ be the stabiliser of $\theta$ under the action of $H$. Now let $\psi$ be an irreducible character of $K$. Then we can define the character $\theta \cdot \psi$ of $A \rtimes K$ by

$$(\theta \cdot \psi)(ak) = \theta(a)\psi(k) \quad \text{for all } a \in A, k \in K.$$

Finally, we induce $\theta \cdot \psi$ to a character $\chi_{\theta,\psi}$ of $A \rtimes H$, so that

$$\chi_{\theta,\psi} = (\theta \cdot \psi) \uparrow_{A \rtimes K}^{A \rtimes H} .$$

We have the following theorem.

**Theorem 5.2.4** (Serre [57, Prop. 25])**.** *With the notation as above, $\chi_{\theta,\psi}$ is an irreducible character of $A \rtimes H$ and every irreducible character of $A \rtimes H$ is of this form. Moreover, if $\chi_{\theta,\psi} = \chi_{\theta',\psi'}$, then $\psi = \psi'$ and $\theta$ and $\theta'$ are in the same orbit under the action of $H$ on $A'$.*

We now want to apply Theorem 5.2.4 to the group $G \wr S_n = G^n \rtimes S_n$, so we need the characters of $G^n$ and the characters of some subgroups of the symmetric group. Since $G$ is abelian, all representations of $G$ are 1-dimensional and the group of the corresponding characters of $G$ is isomorphic to $G$ itself. We fix an isomorphism $g \mapsto \theta^g$ from $G$ to its character group. Then the irreducible characters of $G$ are the characters $\theta^g$ for $g \in G$.

The characters of $G^n$ are products of characters of $G$. Thus, they are indexed by the tuples $(g_1, \dots, g_n) \in G^n$. The irreducible character $\theta^{(g_1,\dots,g_n)}$ is given by

$$\theta^{(g_1,\dots,g_n)}(x_1, \dots, x_n) = \prod_{i=1}^{n} \theta^{g_i}(x_i) \quad \text{for every } (x_1, \dots, x_n) \in G^n.$$

Now consider an element $(g_1, \dots, g_n) \in G^n$. The symmetric group $S_n$ acts on the characters $\theta^{(g_1,\dots,g_n)}$ by permuting the entries of $(g_1, \dots, g_n)$. For $g \in G$, let $k(g)$ be the number of times that $g$ occurs in $(g_1, \dots, g_n)$. Then $\sum_{g \in G} k(g) = n$ and the stabiliser of the character $\theta^{(g_1,\dots,g_n)}$ under the action of $S_n$ is

$$\mathrm{Stab}(\theta^{(g_1,\dots,g_n)}) = \prod_{g \in G} S_{k(g)}$$

where $S_{k(g)}$ permutes the indices $i \in [n]$ with $g_i = g$. Note that this is a Young subgroup of $S_n$.

The irreducible characters of a factor $S_{k(g)}$ are indexed by the partitions of $k(g)$ and denoted by $\psi^\mu$ where $\mu \vdash k(g)$ is a partition. Now, the irreducible characters of $\mathrm{Stab}(\theta^{(g_1,\dots,g_n)})$ are given by

$$\psi^{\underline{\mu}} = \bigotimes_{g \in G} \psi^{\underline{\mu}(g)},$$

where $\underline{\mu} : G \to \mathrm{Par}$ is a function such that $\underline{\mu}(g)$ is a partition of $k(g)$ for every $g \in G$. Combining $\psi^{\underline{\mu}}$ with the character $\theta^{(g_1,\dots,g_n)}$ as in Theorem 5.2.4 gives an irreducible character of $G \wr S_n$. Since we only need the orbit of $\theta^{(g_1,\dots,g_n)}$, so the numbers $k(g)$ for $g \in G$, all information needed is encoded in $\underline{\mu}$ (because we have $k(g) = |\underline{\mu}(g)|$). Thus, the irreducible characters of $G \wr S_n$ are indexed by the set of mappings $\underline{\lambda} : G \to \mathrm{Par}$ such that

$$\sum_{g \in G} |\underline{\lambda}(g)| = n.$$

This is the set $\Lambda_n(G)$ that we have seen earlier. For $\underline{\lambda} \in \Lambda_n(G)$, the corresponding irreducible character is denoted by $\chi^{\underline{\lambda}}$.

**Example 5.2.5.** Let us compute the function $\underline{\lambda}$ indexing the trivial character of $G \wr S_n$. We start by picking the trivial character of $G^n$. The trivial character of $G$ is indexed by 0 and thus the trivial character of $G^n$ is indexed by the tuple $(0, \dots, 0)$. Hence, $\theta^{(0,\dots,0)}$ is the trivial character of $G^n$.

Counting how often the elements of $G$ appear in the tuple, we find $k(0) = n$ and $k(g) = 0$ for every $g \in G$ with $g \neq 0$. This implies that the function $\underline{\lambda}$ indexing the character we are constructing must satisfy $\underline{\lambda}(0) = \lambda$ for some partition $\lambda \vdash n$ and $\underline{\lambda}(g) = \varnothing$ for all $g \in G$ with $g \neq 0$.

Now, we have to pick an irreducible character of the stabiliser of $\theta^{(0,\dots,0)}$ which is the symmetric group $S_n$. In order to obtain the trivial character of $G \wr S_n$, we pick the trivial character $1_{S_n}$ which is indexed by the partition $(n)$. It follows that the function $\underline{\lambda}$ indexing the trivial character is the function given by $\underline{\lambda}(0) = (n)$ and $\underline{\lambda}(g) = \varnothing$ for all $g \in G$ with $g \neq 0$.

Next, we present a different way to write down the irreducible characters of $G \wr S_n$. For a character $\theta$ of $G$ and a character $\psi$ of $S_k$, we define $\theta \wr \psi$ to be the character of $G \wr S_k$ given by

$$(\theta \wr \psi)((g_1, \dots, g_k), \pi) = \theta(g_1) \cdots \theta(g_k)\, \psi(\pi)$$

for all $(g_1, \dots, g_k) \in G^k$ and all $\pi \in S_k$. Now consider an element $\underline{\lambda} \in \Lambda_n(G)$. Then we have partitions $\underline{\lambda}(g)$ for every $g \in G$. The character $\chi^{\underline{\lambda}}$ of $G \wr S_n$ is given by

$$\chi^{\underline{\lambda}} = \bigotimes_{g \in G} \left( \theta^g \wr \psi^{\underline{\lambda}(g)} \right) \uparrow_{G \wr S}^{G \wr S_n}, \tag{5.2}$$

where

$$S = \prod_{g \in G} S_{|\underline{\lambda}(g)|} \tag{5.3}$$

is a Young subgroup of $S_n$ and $G \wr S$ denotes the subgroup $G^n \rtimes S$ of $G \wr S_n = G^n \rtimes S_n$.

**Remark 5.2.6.** Note that we did not specify a tuple $(g_1, \dots, g_n) \in G^n$ as before. The element $\underline{\lambda} \in \Lambda_n(G)$ defines a tuple $(g_1, \dots, g_n)$ up to a permutation of the entries (the element $g \in G$ appears $|\underline{\lambda}(g)|$ times). Different choices of tuples lead to conjugate Young subgroups $S$ and ultimately to the same character $\chi^{\underline{\lambda}}$.

In most cases, it is not important how exactly the group $S$ lies inside the group $S_n$. For example, $S_{\{1,2\}} \times S_{\{3,4\}}$ and $S_{\{1,4\}} \times S_{\{2,3\}}$ are different subgroups of $S_4$, but they are conjugate to each other. The first subgroup corresponds to the ordered set partition $(\{1,2\},\{3,4\})$ while the latter corresponds to the ordered set partition $(\{1,4\},\{2,3\})$. Every time we write down a Young subgroup $S$, it implicitly comes with an ordered set partition $P$. In our case, the parts of $P$ are indexed by the group $G$. The factor $S_{|\underline{\lambda}(g)|}$ permutes the set of indices $i \in [n]$ of the tuple $(g_1, \dots, g_n)$ such that $g_i = g$. This set of indices is also the part of $P$ that is indexed by $g \in G$.

If $\underline{\lambda}(g) = \varnothing$ for some $g \in G$, then the corresponding factor in the product in (5.2) or (5.3) is trivial. The reader should interpret equation (5.2) as first choosing for each $g \in G$ how often each character $\theta^g$ appears, then choosing a partition $\underline{\lambda}(g)$ of the corresponding size to get a character $\psi^{\underline{\lambda}(g)}$, and then combining all these characters.

**Example 5.2.7.** Using the interpretation above, we obtain the trivial character of $G \wr S_n$ by specifying that the trivial character $\theta^0$ of $G$ appears $n$ times and then picking for $\underline{\lambda}(0)$ the partition $(n)$ that indexes the trivial character of $S_n$.

Let us consider the group $C_3 \wr S_6$ for an explicit example where we write $C_3 = \{0,1,2\}$. If we start with the function $\underline{\lambda}$ given by

$$\underline{\lambda}(0) = (2),\ \underline{\lambda}(1) = (31),\ \underline{\lambda}(2) = \varnothing,$$

then we are considering the characters $\theta^0$ and $\theta^1$ of $C_3$ together with the characters $\psi^{(2)}$ and $\psi^{(31)}$ of subgroups of $S_6$. Combining the characters correctly, that is forming

$$\theta^0 \wr \psi^{(2)} = \theta^{(0,0)} \cdot \psi^{(2)} \quad \text{and} \quad \theta^1 \wr \psi^{(31)} = \theta^{(1,1,1,1)} \cdot \psi^{(31)},$$

we obtain a character $\left(\theta^0 \wr \psi^{(2)}\right) \cdot \left(\theta^1 \wr \psi^{(31)}\right)$ of $C_3 \wr (S_2 \times S_4)$. Inducing this character to the group $C_3 \wr S_6$ yields the irreducible character indexed by $\underline{\lambda}$.

In view of Theorem 5.2.4, the $\chi^{\underline{\lambda}}$ are all the irreducible characters of $G \wr S_n$.

## 5.3 NOTIONS OF TRANSITIVITY

We continue to consider the group $W = G \wr S_n$ for a fixed abelian group $G$. We now focus on the different types of transitivity in $W$, which means the different $\Omega$ in Theorem 5.1.4.

### 5.3.1 *Transitivity Types*

We seek a generalisation of $\lambda$-transitivity in the symmetric group. It turns out that we can work with $\lambda$-tabloids, that is ordered set partitions, where we additionally allow colouring the elements of parts of the set partition.

Let $S(G)$ be the set of subgroups of $G$. For a mapping $\underline{\sigma} : S(G) \to \mathrm{Par}$, we define

$$\|\underline{\sigma}\| = \sum_{U \leqslant G} |\underline{\sigma}(U)|.$$

We write

$$\Sigma_n(G) = \{\underline{\sigma} : S(G) \to \mathrm{Par} \mid \|\underline{\sigma}\| = n\}$$

for the set of all mappings $\underline{\sigma} : S(G) \to \mathrm{Par}$ satisfying $\|\underline{\sigma}\| = n$. In what follows, an element $\underline{\sigma} \in \Sigma_n(G)$ will be associated with a subgroup of $W$. Hence, every $\underline{\sigma}$ will represent a certain notion of transitivity (for example edge- and face-transitivity for the 3-dimensional cube described in Chapter 1).

Each $\underline{\sigma} \in \Sigma_n(G)$ can be represented by an ordered set of Young diagrams $Y_U$, one for each subgroup $U \subseteq G$, such that the total number of boxes equals $n$. We define a *$\underline{\sigma}$-tableau* to be a filling of the boxes with pairs $(c, i)$, where $c \in G$ and $i \in [n]$ such that all numbers $i \in [n]$ appearing in the pairs $(c, i)$ are pairwise distinct.

Note that $W$ acts on a filled Young diagram by letting $w \in W$ act on the fillings $(c, i)$ of the boxes. Consider a Young diagram $Y_U$ indexed by a subgroup $U$ and one of its rows $R$ of length $\ell$. Then the group $U \wr S_\ell$ acts on the row $R$ by permuting the fillings $(c, i)$ and changing the elements $c$ using elements of $U$. The partition $\underline{\sigma}(U)$ contains the lengths of the rows of $Y_U$, so the group

$$U \wr S_{\underline{\sigma}(U)} \cong U \wr S_{\underline{\sigma}(U)_1} \times U \wr S_{\underline{\sigma}(U)_2} \times \dots$$

acts on a filled Young diagram $Y_U$ in the same way. We call two filled Young diagrams indexed by $U$ *row-equivalent* if they are in the same orbit under this action. That is, the corresponding rows of the Young diagrams contain the same pairs $(c, i)$ up to permutation and multiplication by elements of $U$. Hence, we can also think of the colours $c$ appearing in the filling of the Young diagrams associated with $U$ as being cosets of $U$.

Two $\underline{\sigma}$-tableaux are *row-equivalent* if all of their Young diagrams $Y_U$ are row-equivalent for every subgroup $U$ of $G$. This induces an equivalence relation on $\underline{\sigma}$-tableaux. A *$\underline{\sigma}$-tabloid* is an equivalence class of this relation.

**Example 5.3.1.** Consider $W = C_4 \wr S_6$ where we identify $C_4$ with $\{0, 1, 2, 3\}$ in the natural way. Then the subgroups of $C_4$ are $\{0\}$, $\{0, 2\}$ and $\{0, 1, 2, 3\}$. Hence, every element $\underline{\sigma} \in \Sigma_6(C_4)$ consists of three Young diagrams, one for each subgroup of $C_4$, such that the total number of boxes is 6. Consider $\underline{\sigma} \in \Sigma_6(C_4)$ given by

$$\begin{array}{ccc} \{0\} & \{0,2\} & \{0,1,2,3\} \\ \begin{array}{|c|c|}\hline \phantom{0} & \phantom{0} \\ \hline \phantom{0} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} & \begin{array}{|c|c|c|}\hline \phantom{0} & \phantom{0} & \phantom{0} \\ \hline\end{array} & \varnothing \end{array} .$$

A possible $\underline{\sigma}$-tableau is given by the following filling of the boxes (where we write $c, i$ for the filling $(c, i)$ for better readability)

$$\begin{array}{ccc} \{0\} & \{0,2\} & \{0,1,2,3\} \\ \begin{array}{|c|c|}\hline 1,6 & 0,4 \\ \hline 3,2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} & \begin{array}{|c|c|c|}\hline 2,3 & 0,1 & 3,5 \\ \hline\end{array} & \varnothing \end{array}$$

where we think of a pair $(c, i)$ as the number $i \in \{1, 2, 3, 4, 5, 6\}$ coloured with the colour $c \in \{0, 1, 2, 3\}$. The corresponding $\underline{\sigma}$-tabloid is given by

$$\begin{array}{ccc} \{0\} & \{0,2\} & \{0,1,2,3\} \\ \begin{array}{|cc|}\hline 1,6 & 0,4 \\ \hline \multicolumn{1}{|c|}{3,2} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} & \begin{array}{|ccc|}\hline 2,3 & 0,1 & 3,5 \\ \hline\end{array} & \varnothing \end{array}$$

where the order of the elements in a row does not matter any more. Furthermore, the $\underline{\sigma}$-tabloid can also be written as

$$\begin{array}{ccc} \{0\} & \{0,2\} & \{0,1,2,3\} \\ \begin{array}{|cc|}\hline 0,4 & 1,6 \\ \hline \multicolumn{1}{|c|}{3,2} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} & \begin{array}{|ccc|}\hline 0,1 & 0,3 & 1,5 \\ \hline\end{array} & \varnothing \end{array}$$

as the fillings only differ by a permutation of the elements in a row and colour changes using the elements of $U$ in the Young diagram associated with $U$. Hence, we can think of the colours in the diagram associated with $U$ as being elements of $C_4/U$.

Note that $W$ acts on the set of $\underline{\sigma}$-tabloids. We define a notion of transitivity using $\underline{\sigma}$-tabloids.

**Definition 5.3.2.** A subset $Y$ of $W$ is called *$\underline{\sigma}$-transitive* if $Y$ is transitive on the set of $\underline{\sigma}$-tabloids.

The stabiliser of a $\underline{\sigma}$-tabloid is obtained by stabilising every row of every diagram. In the example above, the stabiliser is given by

$$\big(\{0\} \wr S_{\{4,6\}} \times \{0\} \wr S_{\{2\}}\big) \times \{0,2\} \wr S_{\{1,3,5\}} \cong (S_2 \times S_1) \times C_2 \wr S_3.$$

In general, the stabiliser of the tableau associated with $U$ is $U \wr S_{\underline{\sigma}(U)}$. Hence, the stabiliser of a $\underline{\sigma}$-tabloid is given by

$$H_{\underline{\sigma}} = \prod_{U \leqslant G} \big(U \wr S_{\underline{\sigma}(U)}\big) \cong \prod_{U \leqslant G} \big(U \wr S_{\underline{\sigma}(U)_1} \times U \wr S_{\underline{\sigma}(U)_2} \times \cdots\big)$$

where $S_{\underline{\sigma}(U)}$ denotes the Young subgroup associated with the partition $\underline{\sigma}(U)$. If $\underline{\sigma}(U) = \varnothing$ for some subgroup $U$, then $U \wr S_{\underline{\sigma}(U)}$ is understood to be the trivial group. We write $\xi^{\underline{\sigma}}$ for the permutation character $\xi_{H_{\underline{\sigma}}}$.

By Lemma 5.1.2, a subset $Y$ of $W$ is *$\underline{\sigma}$-transitive* if and only if $Y$ is transitive on the cosets $W/H_{\underline{\sigma}}$. Since the subgroups $H_{\underline{\sigma}}$ involved are conjugate to each other, all notions of transitivity given by permuting the parts in a partition $\underline{\sigma}(U)$ are equivalent. Thus, we do not get more notions of transitivity by considering compositions instead of partitions in the definition of the set $\Sigma_n(G)$. We can assume that $\underline{\sigma}(U)$ is a partition for every subgroup $U \leqslant G$.

If $G$ is the group of one element, then $\underline{\sigma} \in \Sigma_n(G)$ is given by a partition $\sigma$ of $n$ via $\underline{\sigma}(G) = \sigma$ and the notion of a $\underline{\sigma}$-transitive set coincides with that of a $\sigma$-transitive subset of $S_n$ studied in [46].

We also emphasise the important special case that arises for $|G| > 1$ when $\underline{\sigma} \in \Sigma_n(G)$ is given by $\underline{\sigma}(\{0\}) = (1^t)$ and $\underline{\sigma}(G) = (n-t)$. The corresponding subgroup is $G \wr S_{n-t}$ which gives rise to our analogue of $t$-transitivity for the group $W$. A $\underline{\sigma}$-transitive subset of $W$ with $\underline{\sigma}$ defined as above is transitive on the set of $t$-tuples $((c_1, i_1), \dots, (c_t, i_t))$, where $i_1, i_2, \dots, i_t \in [n]$ are pairwise distinct integers and $c_1, c_2, \dots, c_t \in G$. We can also think of them as $(\sigma, k)$-flags from Chapter 1 with $\sigma = (1^t)$ and $k = n - t$. For $G = C_r$, we shall study these subsets in more detail in Section 5.6.

We can also give a geometric interpretation of some transitivity types using the $n$-dimensional cube.

**Example 5.3.3.** Let $G = C_2$ and $n \in \mathbb{N}$ so that $W$ is the symmetry group of the $n$-dimensional cube. As the group $C_2$ only has two subgroups, we can write the transitivity types $\underline{\sigma}$ as tuples $\underline{\sigma} = (\sigma_1, \sigma_2)$ where $\sigma_1 = \underline{\sigma}(\{0\})$ and $\sigma_2 = \underline{\sigma}(C_2)$. Consider $\underline{\sigma} \in \Sigma_n(G)$ given by $\underline{\sigma} = (t, n-t)$ where we write $(t, n-t)$ for the partition tuple $((t), (n-t))$ for better readability. Then the stabiliser of a $\underline{\sigma}$-tabloid is the subgroup

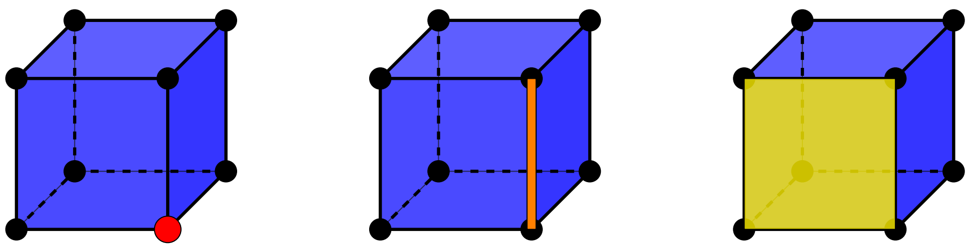
$$H_t = S_t \times C_2 \wr S_{n-t}.$$

The group $H_t$ is also the stabiliser of an $(n-t)$-dimensional face of the cube. To see this, notice that one face of dimension $n - t$ is given by the set of vertices of the form $(1,\dots,1,\pm 1,\dots,\pm 1)$ with precisely $t$ 1s in the front. This set is stabilised by the group

$$S_{\{1,\dots,t\}} \times C_2 \wr S_{\{t+1,\dots,n\}} \cong S_t \times C_2 \wr S_{n-t}$$

and all other stabilisers are conjugate to this group. Thus, a subset $Y$ of $W$ is $(t, n-t)$-transitive if and only if $Y$ is transitive on the $(n-t)$-dimensional faces of the $n$-dimensional cube.

The transitivity types of the form $\underline{\sigma} = (t, n-t)$ for $n = 3$ are visualised in Figure 5.2.



$\underline{\sigma} = (3,0)$ $\underline{\sigma} = (2,1)$ $\underline{\sigma} = (1,2)$

Figure 5.2: Transitivity types in $C_2 \wr S_3$

We can also give an interpretation for the hyperoctahedron because it is dual to the hypercube. If we view $C_2 \wr S_n$ as the symmetry group of the hyperoctahedron, then the subgroup

$$H_t = S_t \times C_2 \wr S_{n-t}$$

is the stabiliser of a $(t-1)$-dimensional face of the hyperoctahedron. Thus, a subset $Y$ of $W$ is $(t, n-t)$-transitive if and only if it is transitive on the $(t-1)$-dimensional faces of the hyperoctahedron.

The following properties of the subgroup $H_{\underline{\sigma}}$ will be helpful.

**Lemma 5.3.4.** *Let $\underline{\sigma} \in \Sigma_n(G)$ and write $H = H_{\underline{\sigma}}$. Then:*

(a) $H \cap G^n = \prod_{U \leqslant G} U^{|\underline{\sigma}(U)|}$

(b) $H \cap S_n = \prod_{U \leqslant G} S_{\underline{\sigma}(U)}$

(c) $(H \cap G^n) \cap (H \cap S_n) = \{\mathrm{id}\}$

(d) $H = (H \cap G^n)(H \cap S_n)$

*Proof.* Parts (a) and (b) follow immediately from the fact that $H_{\underline{\sigma}}$ is a direct product of wreath products on disjoint subsets of $[n]$. Intersecting $H$ with the base group gives the direct product of the base groups of the factors. Part (c) follows because $G^n \cap S_n = \{\mathrm{id}\}$. For (d), notice that $H \cap G^n$ and $H \cap S_n$ are subgroups of $H$ with trivial intersection. Since $H$ is a finite group, the result follows because $|H \cap G^n||H \cap S_n| = |H|$. □

**Example 5.3.5.** Consider $C_4 = \{0, 1, 2, 3\}$ with its subgroups $C_1 = \{0\}$, $C_2 = \{0, 2\}$ and $C_4$. Consider $\underline{\sigma} \in \Sigma_9(C_4)$ given by

$$\underline{\sigma}(C_1) = (31),\ \underline{\sigma}(C_2) = (22),\ \underline{\sigma}(C_4) = (1).$$

Then

$$\begin{aligned} H_{\underline{\sigma}} &\cong S_3 \times S_1 \times C_2 \wr (S_2 \times S_2) \times C_4 \wr S_1 \\ &\cong S_3 \times S_1 \times (C_2^2 \rtimes S_2) \times (C_2^2 \rtimes S_2) \times C_4. \end{aligned}$$

We find

$$H_{\underline{\sigma}} \cap C_4^9 \cong C_1^3 \times C_1 \times C_2^2 \times C_2^2 \times C_4$$

and

$$H_{\underline{\sigma}} \cap S_9 \cong S_3 \times S_1 \times S_2 \times S_2 \times S_1.$$

It follows that

$$H_{\underline{\sigma}} = (C_1^3 \times C_1 \times C_2^2 \times C_2^2 \times C_4)(S_3 \times S_1 \times S_2 \times S_2 \times S_1)$$

if the subgroups are interpreted properly (that is have the copies of the symmetric group act on disjoint subsets and the cyclic groups act on the corresponding entries).

### 5.3.2 *Decomposition of the Permutation Characters*

We now show how $\underline{\sigma}$-transitive subsets of $W$ can be characterised as designs in an association scheme.

For decomposing the permutation characters $\xi^{\underline{\sigma}}$, a certain algorithmically defined relation is helpful. A similar approach for decomposing permutation characters of the hyperoctahedral group $B_n$ (the case $G = C_2$) appears in [49], stated in a very restricted manner. We adapt this approach to wreath products and generalise it to cover all finite abelian groups $G$.

**Remark 5.3.6.** Let us pause for a moment and carefully go through the setup because the notation can become overwhelming quickly.

We have two index sets that we are working with, the set $\Sigma_n(G)$ and the set $\Lambda_n(G)$. The first one indexes the 'types of transitivity' (where we are free to choose how we call the types) and the second one indexes the irreducible characters (where the index set is essentially forced on us by the group structure). Our goal is to find a bijection between transitivity types and sets of irreducible characters. This bijection takes as input a permutation character (in other words, a transitivity type) and outputs the set of its irreducible constituents (a set of irreducible characters). We do not get to choose this bijection. It is the function that outputs the subset $I$ in Theorem 5.1.4. In other words, it tells us the subset $T$ such that the structures we are interested in are Delsarte $T$-designs.

In the results by Martin and Sagan in the symmetric group (see Chapter 4), both the transitivity types ($\lambda$-transitivity) and the irreducible characters $\chi^\lambda$ are indexed by the partitions of $n$. It is easy to forget that there is a fundamental difference between the two index sets. However, for the generalised symmetric group, it is crucial to think of them as two separate objects. Essentially, Young's rule (2.2) gives us the bijection between transitivity types and sets of partitions in the symmetric group.

Coming back to the generalised symmetric group, we have the sets $\Sigma_n(G)$ and $\Lambda_n(G)$. The set $\Sigma_n(G)$ indexes the transitivity types. Some of them have a nice geometric interpretation on regular polytopes. In general, an element $\underline{\sigma} \in \Sigma_n(G)$ defines a $\underline{\sigma}$-tabloid involving coloured set partitions and a subset $Y$ of $W$ is $\underline{\sigma}$-transitive if it is transitive on $\underline{\sigma}$-tabloids. It can be helpful to stay inside the symmetry group of the hypercube $C_2 \wr S_n$ when reading the results and think of the elements of $\Sigma_n(G)$ as defining substructures of the polytope, giving a geometric interpretation of the transitivity type $\underline{\sigma}$.

On the other hand, the set $\Lambda_n(G)$ indexes the irreducible characters of the wreath product $G \wr S_n$. We can think of an element $\underline{\lambda} \in \Lambda_n(G)$ as a tuple with $|G|$ entries where each entry is a partition. In contrast, we can think of the elements of $\Sigma_n(G)$ as tuples with one entry per subgroup of $G$ where each entry is a partition.

We are given a transitivity type $\underline{\sigma} \in \Sigma_n(G)$ and we need to figure out the set of irreducible constituents of the permutation character $\xi^{\underline{\sigma}}$. In other words, we are looking for the corresponding subset of $\Lambda_n(G)$.

We now define a relation on $\Sigma_n(G) \times \Lambda_n(G)$ that gives us the bijection that we need. For a subgroup $U$ of $G$, we write

$$U^\circ = \{g \in G \mid \theta^g(u) = 1 \text{ for all } u \in U\},$$

so $U^\circ$ consists of the elements of $G$ that correspond to irreducible characters that are trivial on $U$.

**Definition 5.3.7.** For $\underline{\sigma} \in \Sigma_n(G)$ and $\underline{\lambda} \in \Lambda_n(G)$, we write

$$\underline{\sigma} \to \underline{\lambda}$$

if there exists $\underline{\mu} \in \Lambda_n(G)$ such that $\underline{\mu}(g) \trianglelefteq \underline{\lambda}(g)$ for each $g \in G$ and $\underline{\mu}$ can be obtained from $\underline{\sigma}$ as follows.

(1) For each $U \leqslant G$, colour each box of the Young diagram of $\underline{\sigma}(U)$ with an element of $U^\circ$.

(2) The element $\underline{\mu} \in \Lambda_n(G)$ arises from the colouring from step (1) in the following way:
For each $g \in G$, consider only the boxes coloured with $g$. Each row of each partition $\underline{\sigma}(U)$ with exactly $k$ boxes coloured with $g$ becomes a part of size $k$ in $\underline{\mu}(g)$ (the actual position of the elements in a row does not matter, only how often $g$ was placed there). In other words, $\underline{\mu}(g)$ has a part of size $k$ for every row of a Young diagram $\underline{\sigma}(U)$ with exactly $k$ appearances of $g$.

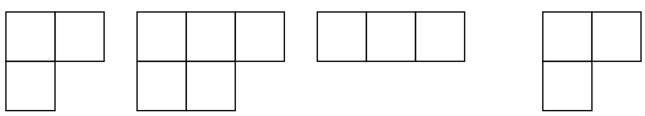
It is important to note that $\{0\}^\circ = G$ and $G^\circ = \{0\}$. This means that the boxes in the Young diagram $\underline{\sigma}(\{0\})$ can be coloured with every element of $G$ while the boxes in the Young diagram $\underline{\sigma}(G)$ all have to be coloured with the element $0$.

**Example 5.3.8.** We take $G = C_6$ and identify $C_6$ with $\{0,1,2,3,4,5\}$ in the natural way. Then the subgroups of $C_6$ are $\{0\}$, $\{0,3\}$, $\{0,2,4\}$, and $\{0,1,2,3,4,5\}$. We get

$$\begin{aligned} \{0\}^\circ &= \{0,1,2,3,4,5\}, \\ \{0,3\}^\circ &= \{0,2,4\}, \\ \{0,2,4\}^\circ &= \{0,3\}, \\ \{0,1,2,3,4,5\}^\circ &= \{0\}. \end{aligned}$$

These are the elements we can use in the colouring of the Young diagrams in step (1) of Definition 5.3.7. Consider $\underline{\sigma} \in \Sigma_{14}(C_6)$ given by

$\{0\}$ $\{0,3\}$ $\{0,2,4\}$ $\{0,1,2,3,4,5\}$



.

A possible colouring of $\underline{\sigma}$ is given by

$$\begin{array}{cccc} \{0\} & \{0,3\} & \{0,2,4\} & \{0,1,2,3,4,5\} \\ \begin{ytableau} 2 & 5 \\ 5 \end{ytableau} & \begin{ytableau} 4 & 2 & 2 \\ 4 & 4 \end{ytableau} & \begin{ytableau} 3 & 0 & 3 \end{ytableau} & \begin{ytableau} 0 & 0 \\ 0 \end{ytableau} \end{array} .$$

Then the corresponding $\underline{\mu} \in \Lambda_{14}(C_6)$ is given by

$$\begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \\ \ydiagram{2,1,1} & \varnothing & \ydiagram{2,1} & \ydiagram{2} & \ydiagram{2,1} & \ydiagram{1,1} \end{array} .$$

This shows that $\underline{\sigma} \to \underline{\mu}$ where

$$\underline{\mu}(0) = (211),\ \underline{\mu}(1) = \varnothing,\ \underline{\mu}(2) = (21),\ \underline{\mu}(3) = (2),\ \underline{\mu}(4) = (21),\ \underline{\mu}(5) = (11).$$

It also shows that $\underline{\sigma} \to \underline{\lambda}$ for every $\underline{\lambda} \in \Lambda_n(G)$ with $\underline{\mu}(g) \trianglelefteq \underline{\lambda}(g)$ for every $g \in G$. Using the dominance order $\trianglelefteq$ to obtain $\underline{\lambda}$ from $\underline{\mu}$ corresponds to moving up boxes in the Young diagrams of $\underline{\mu}(g)$ for each $g \in G$ independently.

**Remark 5.3.9.** Here is another way to think about the relation $\to$. Start with the Young diagrams $\underline{\sigma}(U)$ and $|G|$ empty containers that are indexed with the elements of $G$ (the tuple of these containers will become the element $\underline{\lambda} \in \Lambda_n(G)$ after the algorithm). For each row of each $\underline{\sigma}(U)$, we have to move the boxes of the row to the containers indexed by the elements of $U^\circ$. Moving $k$ boxes from a row of $\underline{\sigma}(U)$ to a container indexed by $g \in U^\circ$ has the effect of adding a part of size $k$ to the container indexed by $g$. Only the number $k$ matters, not the actual position of the boxes in the row. Importantly, this means that for each row of each $\underline{\sigma}(U)$ and each $g \in U^\circ$, we add either one part to the container indexed by $g$ or none. Each box of $\underline{\sigma}$ must be moved into some container.

After all boxes have been moved, the $|G|$ containers have precisely $n$ boxes in total. Now order all the parts in each container so that they form a partition. This defines an element $\underline{\mu} \in \Lambda_n(G)$. Finally, one has the option to use the dominance order $\trianglelefteq$ for each partition $\underline{\mu}(g)$ independently so that we obtain an element $\underline{\lambda} \in \Lambda_n(G)$ with $\underline{\mu}(g) \trianglelefteq \underline{\lambda}(g)$ for every $g \in G$.

The reason for introducing the relation $\to$ is the following theorem. It says that the relation $\to$ precisely characterises the irreducible characters $\chi^{\underline{\lambda}}$ that appear in the decomposition of the permutation character $\xi^{\underline{\sigma}}$.

**Theorem 5.3.10.** *For each $\underline{\sigma} \in \Sigma_n(G)$ and each $\underline{\lambda} \in \Lambda_n(G)$, we have*

$$\langle \xi^{\underline{\sigma}}, \chi^{\underline{\lambda}} \rangle \neq 0 \iff \underline{\sigma} \to \underline{\lambda}.$$

*Proof.* Let $\underline{\lambda} \in \Lambda_n(G)$. We pick a tuple $(g_1, \ldots, g_n) \in G^n$ such that every $g \in G$ occurs precisely $|\underline{\lambda}(g)|$ times. This defines a Young subgroup

$$S = \prod_{g \in G} S_g$$

where $S_g$ permutes the indices $i \in [n]$ with $g_i = g$. Hence, we have an ordered set partition $P$ of $[n]$ indexed by $G$. The part of $P$ indexed by $g \in G$ is the set of indices $i \in [n]$ with $g_i = g$.

We write $K = G \wr S = G^n \rtimes S$. By Theorem 5.2.4, we have $\chi^{\underline{\lambda}} = (\phi \cdot \rho) \uparrow_K^W$ where $\phi$ is the irreducible character of $G^n$ of type $(g_1, \ldots, g_n)$ and $\rho$ is the irreducible character of $S$ given by

$$\rho = \bigotimes_{g \in G} \chi^{\underline{\lambda}(g)}.$$

Write $H = H_{\underline{\sigma}}$. By Frobenius reciprocity, we have

$$\langle \xi^{\underline{\sigma}}, \chi^{\underline{\lambda}} \rangle = \langle 1 \uparrow_H^W, \chi^{\underline{\lambda}} \rangle = \langle 1 \uparrow_H^W, (\phi \cdot \rho) \uparrow_K^W \rangle = \langle 1 \uparrow_H^W \downarrow_K^W, \phi \cdot \rho \rangle.$$

For $s \in W$, write $H^s = sHs^{-1}$. Let $R$ be a complete set of double coset representatives of $K \backslash W / H$. Then by Mackey's formula (Theorem 2.1.34), we have

$$1 \uparrow_H^W \downarrow_K^W = \sum_{s \in R} 1 \downarrow_{H^s \cap K}^{H^s} \uparrow_{H^s \cap K}^{K} = \sum_{s \in R} 1 \uparrow_{H^s \cap K}^{K},$$

and by Frobenius reciprocity, we have

$$\langle \xi^{\underline{\sigma}}, \chi^{\underline{\lambda}} \rangle = \sum_{s \in R} \langle 1 \uparrow_{H^s \cap K}^{K}, \phi \cdot \rho \rangle = \sum_{s \in R} \langle 1, (\phi \cdot \rho) \downarrow_{H^s \cap K}^{K} \rangle. \tag{5.4}$$

Note that all summands on the right hand side are non-negative as each summand is the multiplicity of the irreducible character 1 in the decomposition of the character $(\phi \cdot \rho) \downarrow_{H^s \cap K}^{K}$. Thus, the total sum is non-zero if and only if at least one summand is non-zero. Since $K = G \wr S$, we have $G^n \subseteq K$, so we may assume that all representatives in $R$ are elements of $S_n$. Hence, (5.4) is non-zero if and only if there exists $s \in S_n$ such that

$$\langle 1, (\phi \cdot \rho) \downarrow_{H^s \cap K}^{K} \rangle \neq 0. \tag{5.5}$$

From Lemma 5.3.4, we find $H^s = (H^s \cap G^n)(H^s \cap S_n)$. Hence, as $K = G^n \rtimes S$, it follows that $H^s \cap K = (H^s \cap G^n)(H^s \cap S)$. Since $\phi$ is a character of $G^n$ and $\rho$ is a character of $S$ and since we have $(H^s \cap G^n)(H^s \cap S) = \{\mathrm{id}\}$, elementary calculations reveal that

$$\langle 1, (\phi \cdot \rho) \downarrow_{H^s \cap K}^{K} \rangle = \langle 1, \phi \downarrow_{H^s \cap G^n}^{G^n} \rangle \cdot \langle 1, \rho \downarrow_{H^s \cap S}^{S} \rangle.$$

Hence, the inner product in (5.5) is non-zero if and only if

$$\left\langle 1, \phi \downarrow^{G^n}_{H^s \cap G^n} \right\rangle \cdot \left\langle 1, \rho \downarrow^{S}_{H^s \cap S} \right\rangle \neq 0. \tag{5.6}$$

The left hand side of (5.6) is non-zero if and only if both factors are non-zero. We focus on the second factor first.

As $H^s$ is conjugate to $H$, it is the stabiliser of a $\underline{\sigma}$-tabloid. Hence, by Lemma 5.3.4, $H^s \cap S_n$ is a Young subgroup of $S_n$ whose factors correspond to the parts of $\underline{\sigma}(U)$ as $U$ ranges over the subgroups of $G$. Thus, there is an ordered set partition $Q$ of $[n]$ corresponding to the Young subgroup $H^s \cap S_n$ where every part is associated with a row of a Young diagram $\underline{\sigma}(U)$ for some subgroup $U$.

Recall that $S$ is a Young subgroup with corresponding ordered set partition $P$ where every part is associated with an element $g \in G$. It follows that $H^s \cap S$ is a Young subgroup where the corresponding ordered set partition consists precisely of the (non-empty) parts $p \cap q$ with $p \in P$ and $q \in Q$. This ordered set partition defines a colouring of $\underline{\sigma}$ in the following way.

For every element $p \cap q$ with $p \in P$ and $q \in Q$, we colour the boxes corresponding to $p \cap q$ in the row associated with $q$ with the element $g$ associated with $p$. Hence, by collecting all parts associated with an element $g \in G$ into a partition $\underline{\mu}(g)$, we get a partition tuple $\underline{\mu} \in \Lambda_n(G)$ with $|\underline{\mu}(g)| = |\underline{\lambda}(g)|$ for all $g \in G$ and

$$H^s \cap S = \prod_{g \in G} S_{\underline{\mu}(g)}. \tag{5.7}$$

In other words, $\underline{\mu}$ describes how the factors $S_g$ of $S$ decompose after intersecting $H^s$ with $S$. Each part of $\underline{\mu}(g)$ corresponds to a row of $\underline{\sigma}(U)$ for some $U \leqslant G$ with exactly that many boxes coloured with $g$. Moreover, we have

$$\begin{aligned} \left\langle 1, \rho \downarrow^{S}_{H^s \cap S} \right\rangle &= \left\langle 1, \bigotimes_{g \in G} \chi^{\underline{\lambda}(g)} \downarrow^{S}_{H^s \cap S} \right\rangle \\ &= \prod_{g \in G} \left\langle 1, \chi^{\underline{\lambda}(g)} \downarrow^{S_g}_{S_{\underline{\mu}(g)}} \right\rangle \\ &= \prod_{g \in G} \left\langle 1 \uparrow^{S_g}_{S_{\underline{\mu}(g)}}, \chi^{\underline{\lambda}(g)} \right\rangle. \end{aligned}$$

The left hand side is non-zero if and only if every factor on the right hand side is non-zero. By Young's rule (2.2), this happens if and only if $\underline{\mu}(g) \trianglelefteq \underline{\lambda}(g)$ for each $g \in G$.

Now we consider the first factor on the left hand side of (5.6). Since $\phi$ is an irreducible character of $G^n$, it is of degree 1 and hence also an irreducible character of $H^s \cap G^n$. Because the trivial character 1 is also an irreducible character of $H^s \cap G^n$, it follows from Theorem 2.1.23 that the inner product is non-zero if and only if $\phi \downarrow^{G^n}_{H^s \cap G^n} = 1$. In other words, $\phi(x) = 1$ for all $x \in H^s \cap G^n$.

Since $H^s$ is conjugate to $H$, it is the stabiliser of a $\underline{\sigma}$-tabloid. Hence, $H^s \cap G^n$ is a direct product of subgroups of $G$ where $U \leqslant G$ occurs $|\underline{\sigma}(U)|$ times (see Lemma 5.3.4). This means that to each $i \in \{1, 2, \ldots, n\}$, we assign a subgroup $U_i$ of $G$ such that $H^s \cap G^n = U_1 \times \cdots \times U_n$. Remember that the character $\phi$ is of type $(g_1, \ldots, g_n)$. Since $\phi$ is trivial on $H^s \cap G^n = U_1 \times \cdots \times U_n$, we find that $(g_1, \ldots, g_n) \in U_1^\circ \times \cdots \times U_n^\circ$. Hence, the left factor in (5.6) is non-zero if and only if all boxes of the Young diagram $\underline{\sigma}(U)$ are coloured with elements of $U^\circ$ for every subgroup $U \leqslant G$.

Having completed the setup, we can now prove the statement of the theorem. For the forward implication, suppose that $\langle \xi^{\underline{\sigma}}, \chi^{\underline{\lambda}} \rangle \neq 0$. Then both factors in (5.6) are non-zero. From (5.7), we obtain a colouring $\underline{\mu} \in \Lambda_n(G)$ of $\underline{\sigma}$. Since the first factor of (5.6) is non-zero, all rows of $\underline{\sigma}(U)$ are coloured with elements of $U^\circ$, corresponding to step (1) in Definition 5.3.7. Every part of $\underline{\mu}(g)$ comes from a row with that many boxes coloured with $g$, corresponding to step (2) in Definition 5.3.7. Since the second factor of (5.6) is non-zero, we have $\underline{\mu}(g) \trianglelefteq \underline{\lambda}(g)$ for every $g \in G$. Thus, we obtain $\underline{\sigma} \to \underline{\lambda}$.

For the reverse direction, assume that $\underline{\sigma} \to \underline{\lambda}$. Then there exists a colouring $\underline{\mu} \in \Lambda_n(G)$ of $\underline{\sigma}$ obtained from steps (1) and (2) in Definition 5.3.7. Now consider the Young subgroup

$$T = \prod_{g \in G} T_g$$

of $S_n$ where each part $T_g$ permutes the boxes of $\underline{\sigma}$ (equivalently elements $i \in [n]$) that are coloured with $g \in G$ in the colouring $\underline{\mu}$. Notice that it follows from the definition of the relation $\to$ that $|\underline{\mu}(g)| = |\underline{\lambda}(g)|$ for every $g \in G$.

Recall that the tuple $(g_1, \ldots, g_n)$ we picked in the beginning of the proof was arbitrary. The only requirement was that the element $g$ appears precisely $|\underline{\lambda}(g)|$ times. Hence, we can rearrange the tuple $(g_1, \ldots, g_n)$ such that $T_g$ permutes precisely the indices $i \in [n]$ with $g_i = g$. We find that all of the calculations we did for $S$ also hold for $T$. In fact, $S$ and $T$ are Young subgroups of the same shape so they are conjugate.

Remember that $H \cap S_n$ is a Young subgroup of $S_n$ whose parts correspond to the parts of $\underline{\sigma}(U)$ for $U \leqslant G$. The group $H$ permutes the boxes in the rows of a Young diagram $\underline{\sigma}(U)$ and the group $T$ permutes all boxes that are coloured with the same element $g \in G$. It follows that the group $H \cap T$ permutes all boxes that are coloured with the same element $g \in G$ in a common row of a Young diagram $\underline{\sigma}(U)$. Hence, we find that $H \cap T$ is a Young subgroup of the form $\prod_{g \in G} S_{\underline{\mu}(g)}$ for our colouring $\underline{\mu}$. Thus, Equation (5.7) holds for $S = T$ and $s = \mathrm{id}$.

Since $\underline{\mu}(g) \trianglelefteq \underline{\lambda}(g)$ for each $g \in G$, the second factor of (5.6) is non-zero for $S = T$ and $s = \mathrm{id}$. Moreover, $\phi$ is trivial on $H \cap G^n$, so the first factor of (5.6) is non-zero for $s = \mathrm{id}$. All in all, we obtain $\langle \xi^{\underline{\sigma}}, \chi^{\underline{\lambda}} \rangle \neq 0$. □

We can use Theorem 5.3.10 to obtain a characterisation of $\underline{\sigma}$-transitive sets in $W$.

**Theorem 5.3.11.** *Let $\underline{\sigma} \in \Sigma_n(G)$ and let $Y$ be a non-empty subset of $W$ with dual distribution $(a'_{\underline{\lambda}})$. Then $Y$ is $\underline{\sigma}$-transitive if and only if*

$$a'_{\underline{\lambda}} = 0 \quad \text{for all } \underline{\lambda} \in \Lambda_n(G) \text{ satisfying } \underline{\sigma} \to \underline{\lambda} \text{ and } \underline{\lambda}(0) \neq (n).$$

*Proof.* Follows immediately from Theorem 5.1.4 and Theorem 5.3.10. □

Recall that the element $\underline{\lambda} \in \Lambda_n(G)$ with $\underline{\lambda}(0) = (n)$ indexes the trivial character. The corresponding entry of the dual distribution is always non-zero, which is why this partition tuple has to be excluded. Note that if we let $G = C_1$, then we have $W = C_1 \wr S_n = S_n$ and $\underline{\sigma}$-transitivity in $W$ becomes $\sigma$-transitivity in $S_n$. In this case, the set $\Lambda_n(C_1)$ is the set of partitions of $n$ and the relation $\to$ becomes the dominance order $\trianglelefteq$. Thus, Theorem 5.3.11 specialises to Theorem 4 in [46].

We will see that Theorem 5.3.11 is crucial in order to derive a generalisation of the Livingstone–Wagner theorem in Section 5.5 and to describe $t$-transitive sets in $C_r \wr S_n$ in Section 5.6.

## 5.4 CLIQUES

In this section, we consider so-called *cliques* in $W$ and discuss their relationship to transitive subsets.

**Definition 5.4.1.** Let $\underline{\sigma} \in \Sigma_n(G)$. A non-empty subset $Y$ of $W$ is called a *$\underline{\sigma}$-clique* if for every pair of distinct elements $x, y \in Y$ there is no $\underline{\sigma}$-tabloid that is fixed by $x^{-1}y$.

We are interested in an algebraic characterisation of $\underline{\sigma}$-cliques as cliques in an association scheme. This comes down to determining the conjugacy classes of the elements in the stabiliser $H_{\underline{\sigma}}$ of a $\underline{\sigma}$-tabloid. Note that all stabilisers of $\underline{\sigma}$-tabloids are conjugate to each other, so the answer does not depend on the specific $\underline{\sigma}$-tabloid chosen. Remember that $H_{\underline{\sigma}}$ is an inner direct product of subgroups of the form $U \wr S_k$ for a subgroup $U \leqslant G$ and a positive integer $k$. The conjugacy classes of the group $U \wr S_k$ are indexed by the set $\Lambda_k(U)$. We interpret an element $\underline{\lambda} \in \Lambda_k(U)$ as an element of $\Lambda_k(G)$ by defining $\underline{\lambda}(g) = \emptyset$ for every $g \in G$ with $g \notin U$. Moreover, for $k < n$ we interpret an element $\underline{\lambda} \in \Lambda_k(G)$ as an element $\widetilde{\underline{\lambda}} \in \Lambda_n(G)$ by defining $\widetilde{\underline{\lambda}}(0) = \underline{\lambda}(0) \cup (1^{n-k})$ and $\widetilde{\underline{\lambda}}(g) = \underline{\lambda}(g)$ for all $g \in G \setminus \{0\}$.

For two elements $\underline{\lambda} \in \Lambda_m(G), \underline{\mu} \in \Lambda_n(G)$, we define the element $\underline{\lambda} \cup \underline{\mu} \in \Lambda_{m+n}(G)$ via $\left(\underline{\lambda} \cup \underline{\mu}\right)(g) = \underline{\lambda}(g) \cup \underline{\mu}(g)$ for every $g \in G$. We can now describe the conjugacy classes of the direct product of two groups of the form $U \wr S_k$.

**Lemma 5.4.2.** *Let $U, V \leqslant G$ be subgroups of $G$ and $k, \ell \in \mathbb{N}$ with $k + \ell \leqslant n$. Consider the subgroup $H = (U \wr S_A) \times (V \wr S_B)$ of $G \wr S_n$ where $A, B \subseteq [n]$ are disjoint subsets with $|A| = k$ and $|B| = \ell$. Then the conjugacy classes of $H$ are indexed by the set*

$$\left\{\underline{\lambda} \cup \underline{\mu} \mid \underline{\lambda} \in \Lambda_k(U),\ \underline{\mu} \in \Lambda_\ell(V)\right\}.$$

*Proof.* Every element of $H$ is the product of an element of $U \wr S_A$ and an element of $V \wr S_B$. Let $x = (f, \sigma) \in U \wr S_A$ and $y = (g, \pi) \in V \wr S_B$. Then their product is given by

$$xy = (f + \sigma g, \sigma\pi).$$

Here, we have $f = (f_1, \ldots, f_n) \in G^n$ with $f_a \in U$ for $a \in A$ and $f_i = 0$ otherwise. Similarly, we have $g = (g_1, \ldots, g_n) \in G^n$ with $g_b \in V$ for $b \in B$ and $g_i = 0$ otherwise. Note that since $A$ and $B$ are disjoint, $\sigma$ acts trivially on $g$ and we find that

$$xy = (f + g, \sigma\pi).$$

Recall that the conjugacy classes of $x$ and $y$ are given by the cycle lengths of $\sigma$ and $\pi$, respectively, together with the signs of the cycles (see Section 5.2.1). Since $S_A$ and $S_B$ permute disjoint subsets of $[n]$, we find that the set of cycle lengths of $\sigma\pi$ is simply the disjoint union of the set of cycle lengths of $\sigma$ and the set of cycle lengths of $\pi$. Moreover, since $A$ and $B$ are disjoint, the cycle signs of $\sigma$ and $\pi$ do not change. Hence, if $x$ is of cycle type $\underline{\lambda}$ and $y$ is of cycle type $\underline{\mu}$, then $xy$ is of cycle type $\underline{\lambda} \cup \underline{\mu}$. The cycle type of $xy$ as an element of $G \wr S_n$ is then obtained by adding cycles of length 1 for all elements of $[n] \setminus (A \cup B)$ (if $k + \ell < n$). This corresponds exactly to how we identified elements of $\Lambda_{k+\ell}(G)$ with elements of $\Lambda_n(G)$, proving the theorem. ◻

Denote by $C(\underline{\sigma})$ the set of all $\underline{\lambda} \in \Lambda_n(G)$ such that there exists an element in $H_{\underline{\sigma}}$ that lies in the conjugacy class $C_{\underline{\lambda}}$ of $W$. Applying Lemma 5.4.2 repeatedly immediately gives the following result.

**Theorem 5.4.3.** *Let $\underline{\sigma} \in \Lambda_n(G)$. Consider a stabiliser $H_{\underline{\sigma}}$ of a $\underline{\sigma}$-tabloid and write*

$$H_{\underline{\sigma}} = H_1 \times \cdots \times H_\ell$$

*for some $\ell \in \mathbb{N}$ where $H_i = U_i \wr S_{k_i}$ for some subgroup $U_i \leqslant G$ and $k_i \in \mathbb{N}$. Then we have*

$$C(\underline{\sigma}) = \left\{ \bigcup_{i=1}^{\ell} \underline{\lambda}_i \mid \underline{\lambda}_i \in \Lambda_{k_i}(U_i) \right\}.$$

The following result should be compared to Theorem 5.3.11 and highlights that the concept of a $\underline{\sigma}$-clique is dual to the concept of $\underline{\sigma}$-transitivity. Recall that the conjugacy class of the identity of $W$ is indexed by the unique element $\underline{\lambda} \in \Lambda_n(G)$ with $\underline{\lambda}(0) = (1^n)$.

**Theorem 5.4.4.** *Let $\underline{\sigma} \in \Sigma_n(G)$ and let $Y$ be a non-empty subset of $W$ with inner distribution $(a_{\underline{\lambda}})$. Then $Y$ is a $\underline{\sigma}$-clique if and only if*

$$a_{\underline{\lambda}} = 0 \quad \textit{for all } \underline{\lambda} \in C(\underline{\sigma}) \textit{ with } \underline{\lambda}(0) \neq (1^n).$$

*Proof.* Let $x, y \in Y$ be distinct elements of $Y$. Then the conjugacy class of $x^{-1}y$ is not that of the identity of $W$. If $a_{\underline{\lambda}} = 0$ for every $\underline{\lambda} \in C(\underline{\sigma})$ with $\underline{\lambda}(0) \neq (1^n)$, then the conjugacy class of $x^{-1}y$ is not indexed by an element $\underline{\lambda} \in C(\underline{\sigma})$ because in view of (3.7), $x^{-1}y \in C_{\underline{\lambda}}$ implies $a_{\underline{\lambda}} > 0$. It follows that $x^{-1}y$ cannot be an element of $H^w = wH_{\underline{\sigma}}w^{-1}$ for any $w \in W$. Since the groups $H^w$ for $w \in W$ are exactly all the stabilisers of $\underline{\sigma}$-tabloids, $x^{-1}y$ does not fix a $\underline{\sigma}$-tabloid and thus $Y$ is a $\underline{\sigma}$-clique.

For the converse direction, note that if $x \in C_{\underline{\lambda}}$ stabilises a $\underline{\sigma}$-tabloid $T$, then $wxw^{-1}$ stabilises the $\underline{\sigma}$-tabloid $wT$. Hence, either all elements of a conjugacy class $C_{\underline{\lambda}}$ stabilise some $\underline{\sigma}$-tabloid or none. Thus, if $Y$ is a $\underline{\sigma}$-clique, then the conjugacy class of the quotient $x^{-1}y$ for distinct $x, y \in Y$ cannot be a conjugacy class of an element of $H_{\underline{\sigma}}$. This shows that $a_{\underline{\lambda}} = 0$ for $\underline{\lambda} \in C(\underline{\sigma})$ with $\underline{\lambda}(0) \neq (1^n)$. □

We now establish a connection between $\underline{\sigma}$-transitive sets and $\underline{\sigma}$-cliques in $W$.

**Theorem 5.4.5.** *Let $\underline{\sigma} \in \Lambda_n(G)$, let $H = H_{\underline{\sigma}}$ be the stabiliser of a $\underline{\sigma}$-tabloid and let $Y$ be a non-empty subset of $W$.*

(a) *If $Y$ is a $\underline{\sigma}$-clique, then $|Y| \leqslant |W|/|H|$ with equality if and only if $Y$ is $\underline{\sigma}$-transitive.*

(b) *If $Y$ is $\underline{\sigma}$-transitive, then $|Y| \geqslant |W|/|H|$ with equality if and only if $Y$ is a $\underline{\sigma}$-clique.*

*In both cases, equality implies that $Y$ is sharply $\underline{\sigma}$-transitive.*

*Proof.* Observe that for each $(x,y) \in H \times Y$ there is precisely one element $w \in W$ with $wx = y$ (namely $w = yx^{-1}$). Thus, we have

$$\sum_{w \in W} |Y \cap wH| = |Y| \cdot |H|. \tag{5.8}$$

The quotient of any two elements in $Y \cap wH$ fixes a $\underline{\sigma}$-tabloid. Hence, if $Y$ is a $\underline{\sigma}$-clique, then every summand on the left hand side of (5.8) is at most 1. This gives $|Y| \cdot |H| \leqslant |W|$, proving the bound in (a). Now, since $H$ is the stabiliser of a $\underline{\sigma}$-tabloid $T$, the coset $wH$ contains precisely the elements of $W$ that map $T$ to $wT$. Thus, if $Y$ is $\underline{\sigma}$-transitive, then every summand on the left hand side of (5.8) is at least 1. This gives $|Y| \cdot |H| \geqslant |W|$, proving the bound in (b).

In both cases, equality occurs if and only if $|Y \cap w\widetilde{H}| = 1$ for every $w \in W$ and every stabiliser $\widetilde{H}$ of a $\underline{\sigma}$-tabloid. This is equivalent to $Y$ being sharply $\underline{\sigma}$-transitive. ☐

Theorem 5.4.5 (a) can also be obtained using the clique-coclique bound (see Theorem 3.3.6).

## 5.5 COMPARISON OF TRANSITIVITY TYPES

We now turn to generalisations of Theorem 4.3.3 by Livingstone and Wagner. Let $\underline{\sigma}, \underline{\tau} \in \Sigma_n(G)$. We write $\underline{\sigma} \succeq \underline{\tau}$ if for every subset $Y$ of $W$ we have

$$Y \text{ is } \underline{\sigma}\text{-transitive} \implies Y \text{ is } \underline{\tau}\text{-transitive}.$$

Hence, writing

$$M_{\underline{\sigma}} = \{\underline{\lambda} \in \Lambda_n(G) \mid \underline{\sigma} \to \underline{\lambda}\},$$

Theorem 5.3.11 implies

$$\underline{\sigma} \succeq \underline{\tau} \iff M_{\underline{\sigma}} \supseteq M_{\underline{\tau}}.$$

We shall give a characterisation of the relation $\succeq$ in special cases. We say that $\underline{\sigma} \in \Sigma_n(G)$ is *simple* if $\underline{\sigma}(U) = \varnothing$ for each non-trivial proper subgroup $U$ of $G$. If $|G| > 1$, then the *shape* of a simple $\underline{\sigma} \in \Sigma_n(G)$ is the pair $(\sigma, \rho)$ where $\underline{\sigma}(\{0\}) = \sigma$ and $\underline{\sigma}(G) = \rho$. In order to fit the case $|G| = 1$ into this framework, we define the shape of a partition $\sigma \in \Sigma_n(\{0\})$ to be $(\sigma, \varnothing)$. We shall identify a simple $\underline{\sigma} \in \Sigma_n(G)$ with its shape. We define the *size* of a double partition $(\sigma, \rho)$ to be $|\sigma| + |\rho|$. For double partitions $(\sigma, \rho)$ and $(\tau, \pi)$ of the same size $n$, we write $(\sigma, \rho) \succeq (\tau, \pi)$ if their corresponding partition tuples in $\Sigma_n(G)$ are related in this way. Note that if $G$ has prime order, then simplicity of $\underline{\sigma} \in \Sigma_n(G)$ is no restriction.

We call $\underline{\sigma} \in \Sigma_n(G)$ *parabolic* if it is simple of shape $(\sigma, (k))$ for a non-negative integer $k$ (the case $k = 0$ should be read as $(\sigma, \varnothing)$). We write $(\sigma, k)$ for the shape $(\sigma, (k))$. The term "parabolic" stems from the fact that if $\underline{\sigma}$ is parabolic in the hyperoctahedral group, the case $G = C_2$, then the corresponding stabiliser $H_{\underline{\sigma}}$ is a parabolic subgroup if $W$ is viewed as a Coxeter group. It turns out that parabolic $\underline{\sigma}$ are especially nice to work with and capture important transitivity types like $t$-transitivity for $\underline{\sigma} = ((1^t), n-t)$ and $t$-homogeneity for $\underline{\sigma} = ((t), n-t)$.

**Example 5.5.1.** Consider $W = C_4 \wr S_9$ where we write $C_4 = \{0,1,2,3\}$. Then the subgroups of $C_4$ are $C_1 = \{0\}$, $C_2 = \{0,2\}$ and $C_4 = \{0,1,2,3\}$. The element $\underline{\sigma} \in \Sigma_9(C_4)$ with

$$\underline{\sigma}(C_1) = (32),\ \underline{\sigma}(C_2) = (2),\ \underline{\sigma}(C_4) = (11)$$

is not simple because $\underline{\sigma}(C_2) \neq \varnothing$. The element $\underline{\tilde{\sigma}} \in \Sigma_9(C_4)$ with

$$\underline{\tilde{\sigma}}(C_1) = (321),\ \underline{\tilde{\sigma}}(C_2) = \varnothing,\ \underline{\tilde{\sigma}}(C_4) = (21)$$

is simple and its shape is the double partition $\big((321), (21)\big)$. However, $\underline{\tilde{\sigma}}$ is not parabolic because $\underline{\tilde{\sigma}}(C_4)$ has more than one part.

We start with some simple yet helpful properties of the relation $\geq$.

**Proposition 5.5.2.** *Let $(\sigma, \rho)$ and $(\tau, \pi)$ be two double partitions of the same size satisfying $(\sigma, \rho) \geq (\tau, \pi)$. Then we have*

(*a*) $\rho \trianglelefteq \pi$,

(*b*) $\sigma' \trianglerighteq \tau'$,

(*c*) $\sigma \cup \rho \trianglelefteq \tau \cup \pi$.

*Proof.* To prove (a) and (b), consider $\underline{\lambda} \in \Lambda_n(G)$ given by $\underline{\lambda}(0) = \pi$ and $\underline{\lambda}(g) = \tau$ for some fixed non-zero $g \in G$ (in the case $G = \{0\}$, we have $\pi = \varnothing$ and use $\underline{\lambda} \in \Lambda_n(\{0\})$ with $\underline{\lambda}(0) = \tau$). Then we have $(\tau, \pi) \to \underline{\lambda}$, and since $(\sigma, \rho) \geq (\tau, \pi)$, we must also have $(\sigma, \rho) \to \underline{\lambda}$. In other words, there must be a colouring of $(\sigma, \rho)$ giving rise to $\underline{\mu} \in \Lambda_n(G)$ with $\underline{\mu}(g) \trianglelefteq \underline{\lambda}(g)$ for every $g \in G$.

Let $\underline{\mu} \in \Lambda_n(G)$ be obtained from such a colouring of $(\sigma, \rho)$. The parts of $\rho$ must occur as parts in $\underline{\mu}(0)$. Hence, $\rho \subseteq \underline{\mu}(0) \trianglelefteq \underline{\lambda}(0) = \pi$, and so $\rho \trianglelefteq \pi$, proving (a).

Now consider the partition $\underline{\mu}(g)$ for the fixed non-zero $g \in G$ from above. Only boxes of $\sigma$ can be coloured with $g$, so all boxes of $\underline{\mu}(g)$ have to come from $\sigma$. Notice that for every $k$, the number of boxes in the first $k$ columns of $\underline{\mu}(g)$ cannot be larger than the number of boxes in the first $k$ columns of $\sigma$. Hence $\sigma' \trianglerighteq \underline{\mu}(g)' \trianglerighteq \underline{\lambda}(g)' = \tau'$ and so $\sigma' \trianglerighteq \tau'$, proving (b).

To prove (c), consider $\underline{\lambda} \in \Lambda_n(G)$ given by $\underline{\lambda}(0) = \tau \cup \pi$. Then we have $(\tau, \pi) \to \underline{\lambda}$ and so $(\sigma, \rho) \to \underline{\lambda}$. The only way to achieve this is to colour every box of $(\sigma, \rho)$ with the element 0. This forces $\sigma \cup \rho \trianglelefteq \tau \cup \pi$. $\square$

**Example 5.5.3.** The following example shows that the conditions (a), (b), (c) in Proposition 5.5.2 are in general not sufficient to guarantee $(\sigma, \rho) \succeq (\tau, \pi)$. Take

$$(\sigma, \rho) = ((311), (22)) \quad \text{and} \quad (\tau, \pi) = ((22), (311)).$$

Then, the conditions of Proposition 5.5.2 are satisfied.

Let $\underline{\lambda} \in \Lambda_9(G)$ be given by $\underline{\lambda}(0) = (31^4)$ and $\underline{\lambda}(g) = (1^2)$ for some fixed non-zero $g \in G$. Then $(\tau, \pi) \to \underline{\lambda}$ holds, but $(\sigma, \rho) \to \underline{\lambda}$ does not hold. Hence, $(\sigma, \rho) \succeq (\tau, \pi)$ is not true.

Notice however that in the case $G = \{0\}$, $G \wr S_n$ is the symmetric group and all three conditions in Proposition 5.5.2 reduce to $\sigma \trianglelefteq \tau$. This is sufficient to guarantee that a $\sigma$-transitive set is also $\tau$-transitive by Theorem 4.3.7.

We shall however prove that the conditions (a) and (c) in Proposition 5.5.2 are sufficient to guarantee $(\sigma, \rho) \succeq (\tau, \pi)$ in the parabolic case.

Let us first investigate how the relation $\succeq$ behaves when changing the shape $(\sigma, \rho)$ slightly.

**Lemma 5.5.4** (Dominance rule). *Let $(\sigma, \rho)$ and $(\tau, \pi)$ be two double partitions of the same size satisfying $|\sigma| = |\tau|$, $\sigma \trianglelefteq \tau$ and $\rho \trianglelefteq \pi$. Then we have $(\sigma, \rho) \succeq (\tau, \pi)$.*

*Proof.* Since $\succeq$ is a transitive relation, we can show $(\sigma, \rho) \succeq (\tau, \pi)$ in two steps via $(\sigma, \rho) \succeq (\sigma, \pi) \succeq (\tau, \pi)$.

First, we show that $(\sigma, \rho) \succeq (\sigma, \pi)$. Consider $\underline{\lambda} \in \Lambda_n(G)$ with $(\sigma, \pi) \to \underline{\lambda}$. We have to show $(\sigma, \rho) \to \underline{\lambda}$. According to Definition 5.3.7, there is $\underline{\beta} \in \Lambda_n(G)$ arising from a colouring $C$ of the boxes of $(\sigma, \pi)$ such that $\underline{\beta}(g) \trianglelefteq \underline{\lambda}(g)$ for all $g \in G$. We need to find a colouring $\tilde{C}$ of $(\sigma, \rho)$ in step (1) of Definition 5.3.7 such that we obtain $(\sigma, \rho) \to \underline{\lambda}$.

Define $\tilde{C}$ by colouring the boxes of $\sigma$ in $(\sigma, \rho)$ in the same way as in the colouring $C$ of $(\sigma, \pi)$. Colour all boxes of $\rho$ with 0 (this is forced because of $G^\circ = \{0\}$). This assignment defines a colouring $\tilde{C}$ of $(\sigma, \rho)$ which gives $\underline{\gamma} \in \Lambda_n(G)$ with $\underline{\gamma}(g) = \underline{\beta}(g)$ for $g \neq 0$. Moreover, writing $\alpha$ for the partition of boxes of $\sigma$ that are coloured with 0, we have $\underline{\gamma}(0) = \alpha \cup \rho$ and $\underline{\beta}(0) = \alpha \cup \pi$. Since we have $\rho \trianglelefteq \pi$, we obtain $\underline{\gamma}(0) \trianglelefteq \underline{\beta}(0)$. Hence, we find that $\underline{\gamma}(g) \trianglelefteq \underline{\lambda}(g)$ for every $g \in G$, proving $(\sigma, \rho) \to \underline{\lambda}$.

Now we show $(\sigma, \pi) \succeq (\tau, \pi)$. Consider $\underline{\lambda} \in \Lambda_n(G)$ with $(\tau, \pi) \to \underline{\lambda}$. We have to show $(\sigma, \pi) \to \underline{\lambda}$. Since $\sigma \trianglelefteq \tau$, the Young diagram of $\tau$ can be obtained from the Young diagram of $\sigma$ by moving up boxes one at a time. Since the relation $\succeq$ is transitive, it is enough to consider a single box that is moved up, say from row $j$ to row $i$. Thus, we can assume that $\tau_i = \sigma_i + 1$, $\tau_j = \sigma_j - 1$ and $\tau_k = \sigma_k$ for every $k \neq i, j$.

According to Definition 5.3.7, there is $\underline{\beta} \in \Lambda_n(G)$ arising from a colouring of the boxes of $(\tau, \pi)$ such that $\underline{\beta}(g) \trianglelefteq \underline{\lambda}(g)$ for all $g \in G$. Consider the colouring of the boxes of $\tau$ in row $i$ and row $j$. Since $\sigma_i \geqslant \sigma_j$, we have $\tau_i \geqslant \tau_j + 2$. Thus, there exists an element $g \in G$ such that in row $i$ of $\tau$ there are more boxes coloured with $g$ than in row $j$ of $\tau$. Because the actual position of the colours in a row does not matter, we can assume without loss of generality that the boxes at the end of row $i$ are coloured with $g$.

Now define a colouring of $(\sigma, \pi)$ by copying the colouring of the boxes of $\tau$. In other words, the box that was moved from row $j$ to row $i$ to obtain $\tau$ from $\sigma$ is coloured with $g$ and all other boxes of $\sigma$ are coloured in the same way that they were coloured in $\tau$. All boxes of $\pi$ have to be coloured with 0. This colouring of $(\sigma, \pi)$ gives $\underline{\gamma} \in \Lambda_n(G)$ with $\underline{\gamma}(h) = \underline{\beta}(h)$ for every $h \neq g$.

Now consider $\underline{\beta}(g)$ and $\underline{\gamma}(g)$. Let $x$ denote the number of times that $g$ appears in row $j$ of $\tau$ in the colouring giving rise to $\underline{\beta}$. By the choice of $g$, we find that $g$ appears $x + k$ times in row $i$ of $\tau$ for some $k \geqslant 1$. Thus, $\underline{\beta}(g)$ contains the parts $x$ and $x + k$. By construction of the colouring of $(\sigma, \pi)$, $\underline{\gamma}(g)$ contains the parts $x + 1$ and $x + k - 1$. Since all other parts of $\underline{\beta}(g)$ and $\underline{\gamma}(g)$ are the same, we get $\underline{\gamma}(g) \trianglelefteq \underline{\beta}(g)$. Hence, we find that $\underline{\gamma}(g) \trianglelefteq \underline{\lambda}(g)$ for every $g \in G$, proving $(\sigma, \pi) \to \underline{\lambda}$. □

Proposition 5.5.2 and Lemma 5.5.4 immediately give a characterisation of the relation $\succeq$ in the case $|\sigma| = |\tau|$.

**Corollary 5.5.5.** *Let $(\sigma, \rho)$ and $(\tau, \pi)$ be two double partitions of the same size. If $|\sigma| = |\tau|$ (thus $|\rho| = |\pi|$), then*

$$(\sigma, \rho) \succeq (\tau, \pi) \iff \sigma \trianglelefteq \tau \text{ and } \rho \trianglelefteq \pi.$$

If $\sigma$ is a partition and $k$ and $r$ are positive integers, we denote by $\sigma +_r k$ the partition obtained by adding $k$ to the part $\sigma_r$ of $\sigma$ and then possibly reordering the parts to obtain a partition. In other words, we add the sequence $0^{r-1}k$ to the sequence $\sigma$ and reorder $\sigma$ such that it becomes a partition.

The next lemma describes a way to move some boxes from the left entry of a double partition $(\sigma, \rho)$ to the right entry.

**Lemma 5.5.6** (Sum rule)**.** *Let $(\sigma, \rho)$ be a double partition satisfying $\sigma_r \leqslant \rho_s$ for some positive integers $r$ and $s$. Then, for each non-negative integer $k$, we have*

$$(\sigma +_r k, \rho) \succeq (\sigma, \rho +_s k).$$

*Proof.* Let $\underline{\lambda} \in \Lambda_n(G)$ with $(\sigma, \rho +_s k) \to \underline{\lambda}$. We have to show that $(\sigma +_r k, \rho) \to \underline{\lambda}$. According to Definition 5.3.7, there is $\underline{\beta} \in \Lambda_n(G)$ arising from a colouring $C$ of $(\sigma, \rho +_s k)$ such that $\underline{\beta}(g) \trianglelefteq \underline{\lambda}(g)$ for all $g \in G$.

Define a colouring $\widetilde{C}$ of $(\sigma +_r k, \rho)$ as follows. Colour all boxes of $\sigma +_r k$ that are also boxes of $\sigma$ in the same way they are coloured in the colouring $C$ of $(\sigma, \rho +_s k)$. Colour the $k$ remaining boxes (that is the boxes that were added to the part $\sigma_r$) and all boxes of $\rho$ with the element $0$ (the colouring of the boxes of $\rho$ is forced). This colouring $\widetilde{C}$ of $(\sigma +_r k, \rho)$ gives $\underline{\gamma} \in \Lambda_n(G)$ with $\underline{\gamma}(g) = \underline{\beta}(g)$ for all $g \neq 0$.

Note that in the part $\sigma_r + k$, there are $x + k$ boxes coloured with $0 \in G$ for some integer $x$ satisfying $0 \leqslant x \leqslant \sigma_r$. Hence, $\underline{\beta}(0)$ contains the parts $x$ and $\rho_s + k$ while $\underline{\gamma}(0)$ contains the parts $x + k$ and $\rho_s$. We now distinguish the order of these parts.

If $x + k \leqslant \rho_s$, then $(\rho_s, x + k)$ is a partition satisfying $(\rho_s, x + k) \trianglelefteq (\rho_s + k, x)$. If $x + k > \rho_s$, then $(x + k, \rho_s)$ is a partition satisfying $(x + k, \rho_s) \trianglelefteq (\rho_s + k, x)$ since $x \leqslant \sigma_r \leqslant \rho_s$. In both cases, we find that $\underline{\gamma}(0) \trianglelefteq \underline{\beta}(0) \trianglelefteq \underline{\lambda}(0)$ and so $\underline{\gamma}(0) \trianglelefteq \underline{\lambda}(0)$. Hence, we find that $\underline{\gamma}(g) \trianglelefteq \underline{\lambda}(g)$ for every $g \in G$, proving $(\sigma +_r k, \rho) \to \underline{\lambda}$. □

Figure 5.3 illustrates the sum rule.

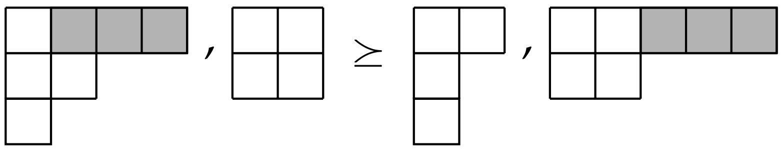


Figure 5.3: The sum rule

Notice that in Figure 5.3, we could have used the sum rule twice to avoid having to reorder the rows of the left tableau. First, use the sum rule to move one box from the second row of the left tableau to the first row of the right tableau. Then, use the sum rule again to move two boxes from the first row of the left tableau to the first row of the right tableau. Figure 5.4 illustrates this.

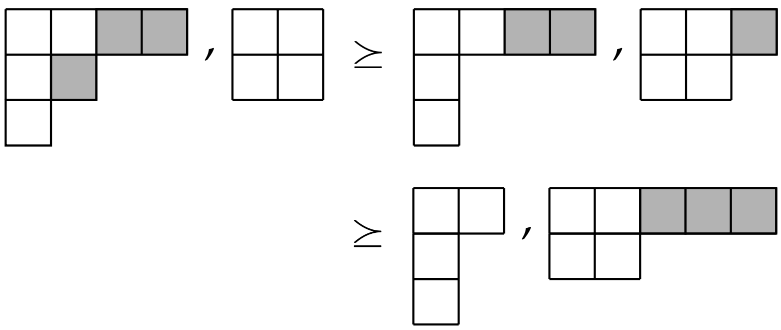


Figure 5.4: The sum rule revisited

We can think of the grey boxes in Figure 5.3 as 'falling down by gravity' so that we pick the grey box in row two in Figure 5.4 rather than the grey box in column two and row one in Figure 5.3. Working our way from the bottom up, it is clear that the sum rule can always be applied and no parts have to be reordered.

We emphasise two special cases of the sum rule. The first rule says that we can move whole parts from the left entry of the double partition $(\sigma, \rho)$ to the right entry.

**Corollary 5.5.7** (Union rule)**.** *Let $(\sigma, \rho)$ be a double partition. Then for every partition $\alpha$ we have*

$$(\sigma \cup \alpha, \rho) \geq (\sigma, \rho \cup \alpha).$$

*Proof.* Since the relation $\geq$ is transitive, it is enough to consider $\alpha = (k)$ for some integer $k$. Take $r$ and $s$ in the sum rule (Lemma 5.5.6) such that $\sigma_r = \rho_s = 0$. Then $\sigma +_r k = \sigma \cup (k)$ and $\rho +_s k = \rho \cup (k)$. □

The second special case states that we can exchange a part from the left entry of the double partition $(\sigma, \rho)$ with a part from the right entry provided that the part on the left is bigger than the part on the right.

**Corollary 5.5.8** (Exchange rule)**.** *Let $(\sigma, \rho)$ be a double partition and let $i, j$ be integers with $\sigma_i > \rho_j$. Define the double partition $(\tilde{\sigma}, \tilde{\rho})$ as the double partition $(\sigma, \rho)$ but with the part $\sigma_i$ of $\sigma$ and the part $\rho_j$ of $\rho$ exchanged (then possibly reordering the parts to obtain a partition). Then we have $(\sigma, \rho) \geq (\tilde{\sigma}, \tilde{\rho})$.*

*Proof.* Write $\sigma_i = \rho_j + (\sigma_i - \rho_j)$ and take $k = \sigma_i - \rho_j$ in the sum rule (Lemma 5.5.6). □

Figures 5.5 and 5.6 visualise the union rule and the exchange rule.

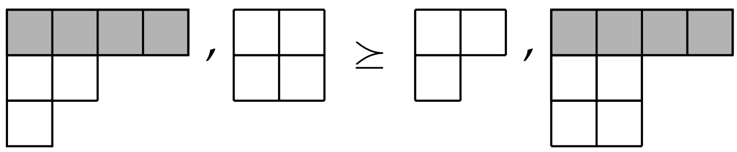


Figure 5.5: The union rule

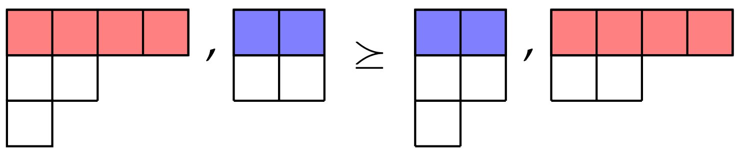


Figure 5.6: The exchange rule

It turns out that using the dominance rule and the sum rule, we can show that the necessary conditions from Proposition 5.5.2 are actually sufficient for parabolic types. Hence, we can completely characterise the relation $\geq$ for parabolic types.

Before we prove this, we define the union of a partition with a sequence of the form $(0^k, \ell - k)$ for some integers $0 \leqslant k \leqslant \ell$. This will be useful to neatly write down some conjugate partitions.

**Definition 5.5.9.** Let $\lambda$ be a partition and let $k, \ell \in \mathbb{N}$ with $0 \leqslant k \leqslant \ell$. Let $i \in \mathbb{N}$ be minimal such that $\lambda_i \leqslant k$ (such an $i$ always exists as $\lambda$ is an infinite sequence ending in zeroes). Then we define $\lambda \cup (0^k, \ell - k)$ to be the partition given by

$$\lambda \cup (0^k, \ell - k) = \begin{cases} \lambda +_i (\ell - k), & \text{if } i = 1 \text{ or } i \geqslant 2 \text{ and } \ell \leqslant \lambda_{i-1}, \\ \lambda +_{i-1} (\ell - \lambda_{i-1}) +_i (\lambda_{i-1} - k), & \text{otherwise.} \end{cases}$$

The operation of adding a part of the form $(0^k, \ell - k)$ to a partition corresponds to adding a box to the columns $k+1, k+2, \ldots, \ell$. If there is a gap between some boxes in a row, then the boxes are moved to the left to close the gap. Figure 5.7 gives two examples using the partitions $(111) \cup (0^2, 3) = (411)$ and $(4311) \cup (0^2, 3) = (5421)$.

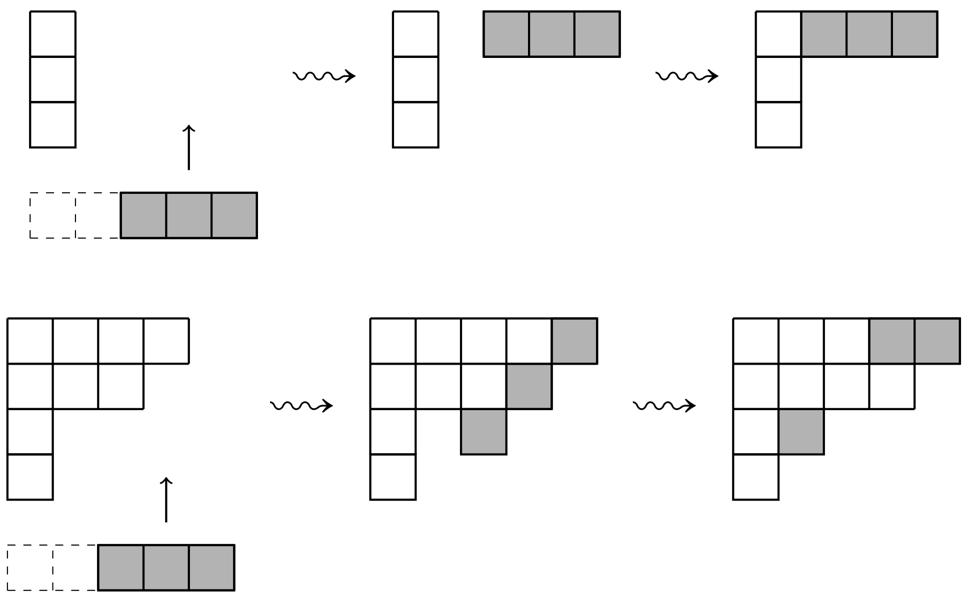


Figure 5.7: The partitions $(111) \cup (0^2, 3)$ and $(4311) \cup (0^2, 3)$

The reason for introducing this operation is the following lemma.

**Lemma 5.5.10.** *Let $n, k, \ell \in \mathbb{N}$ with $0 \leqslant k \leqslant \ell$ and let $\lambda \vdash n$ and $\mu \vdash n - (\ell - k)$ be partitions. Then we have*

$$\lambda \trianglerighteq \mu + 0^k 1^{\ell - k} \implies \lambda' \trianglelefteq \mu' \cup (0^k, \ell - k).$$

*Proof.* Assume that $\lambda \trianglerighteq \mu + 0^k 1^{\ell-k}$ and write $\widetilde{\mu} = \mu + 0^k 1^{\ell-k}$. In general, $\widetilde{\mu}$ is a composition. Notice that adding boxes to the rows of $\mu$ is equivalent to adding boxes to the columns of $\mu'$. Hence, if $\widetilde{\mu}$ is a partition, then we can simply take duals and deduce

$$\lambda' \trianglelefteq \mu' \cup (0^k, \ell - k).$$

We show that this is still true if $\widetilde{\mu}$ is not a partition.

If $\widetilde{\mu}$ is not a partition, then there exists $i \in \mathbb{N}$ with $\widetilde{\mu}_i < \widetilde{\mu}_{i+1}$. For $\mu + 0^k 1^{\ell-k}$, this can only happen for $i = k$ but we will not assume this in order to iterate the argument. We distinguish two cases.

If $\lambda_i < \widetilde{\mu}_{i+1}$, then we find $\lambda_{i+1} \leqslant \lambda_i < \widetilde{\mu}_{i+1}$. Because of $\lambda \trianglerighteq \widetilde{\mu}$, we have

$$\lambda_1 + \ldots + \lambda_i \geqslant \widetilde{\mu}_1 + \ldots + \widetilde{\mu}_i.$$

If this inequality was an equality, then by adding $\lambda_{i+1}$ to both sides it would follow that

$$\lambda_1 + \ldots + \lambda_i + \lambda_{i+1} = \widetilde{\mu}_1 + \ldots + \widetilde{\mu}_i + \lambda_{i+1} < \widetilde{\mu}_1 + \ldots + \widetilde{\mu}_i + \widetilde{\mu}_{i+1},$$

contradicting $\lambda \trianglerighteq \widetilde{\mu}$. Hence, we find

$$\lambda_1 + \ldots + \lambda_i > \widetilde{\mu}_1 + \ldots + \widetilde{\mu}_i \implies \lambda_1 + \ldots + \lambda_i \geqslant \widetilde{\mu}_1 + \ldots + \widetilde{\mu}_i + 1.$$

Now

$$\lambda_1 + \ldots + \lambda_{i+1} \geqslant \widetilde{\mu}_1 + \ldots + \widetilde{\mu}_{i+1} > \widetilde{\mu}_1 + \ldots + \widetilde{\mu}_{i+1} - 1,$$

so $\lambda$ also dominates the composition $(\widetilde{\mu}_1, \ldots, \widetilde{\mu}_i + 1, \widetilde{\mu}_{i+1} - 1, \ldots)$ that is obtained from $\widetilde{\mu}$ by moving a box from row $i+1$ to row $i$.

If $\lambda_i \geqslant \widetilde{\mu}_{i+1}$, then $\lambda_i > \widetilde{\mu}_i$, so $\lambda_i - \widetilde{\mu}_i > 0$. Because of $\lambda \trianglerighteq \widetilde{\mu}$, we find

$$\lambda_1 + \ldots + \lambda_i \geqslant \widetilde{\mu}_1 + \ldots + \widetilde{\mu}_i.$$

If this inequality was an equality, then it would follow that

$$\lambda_1 + \ldots + \lambda_{i-1} < \lambda_1 + \ldots + \lambda_{i-1} + \lambda_i - \widetilde{\mu}_i = \widetilde{\mu}_1 + \ldots + \widetilde{\mu}_i - \widetilde{\mu}_i = \widetilde{\mu}_1 + \ldots + \widetilde{\mu}_{i-1},$$

contradicting $\lambda \trianglerighteq \widetilde{\mu}$. So we again deduce that $\lambda$ also dominates the composition $(\widetilde{\mu}_1, \ldots, \widetilde{\mu}_i + 1, \widetilde{\mu}_{i+1} - 1, \ldots)$ that is obtained from $\widetilde{\mu}$ by moving a box from row $i+1$ to row $i$.

As long as $\widetilde{\mu}$ is not a partition, so as long as we find an index $i$ with $\widetilde{\mu}_i < \widetilde{\mu}_{i+1}$, we can repeat the argument. Applied to the composition $\widetilde{\mu} = \mu + 0^k 1^{\ell-k}$, we find that if $\widetilde{\mu}$ is not a partition, then $\widetilde{\mu}_{k+1} = \widetilde{\mu}_k + 1 > \widetilde{\mu}_k$. Hence, we can move the box at the end of row $k+1$ as far up in its column as it will go. If there are more parts of the same size below, we can repeat the process. In the end, we will have moved all boxes in that column to the top. This creates a partition.

It follows that if $\lambda \trianglerighteq \mu + 0^k 1^{\ell-k}$, then $\lambda$ also dominates the partition that is obtained from $\mu + 0^k 1^{\ell-k}$ by moving all boxes in a column as far up as possible. In the dual, this corresponds to moving all boxes in a row as far left as possible. This agrees exactly with Definition 5.5.9, so we obtain

$$\lambda' \trianglelefteq \mu' \cup (0^k, \ell - k). \qquad \square$$

We can now characterise the relation $\succeq$ for parabolic types.

**Theorem 5.5.11.** *Let $(\sigma, k)$ and $(\tau, \ell)$ be two double partitions of the same size. Then*

$$(\sigma, k) \succeq (\tau, \ell) \iff k \leqslant \ell \text{ and } (\sigma \cup (k)) \trianglelefteq (\tau \cup (\ell)).$$

*Proof.* The forward direction follows from Proposition 5.5.2.

For the reverse direction, assume that $k \leqslant \ell$ and $(\sigma \cup (k)) \trianglelefteq (\tau \cup (\ell))$. Taking duals, it follows that $\sigma' + 1^k \trianglerighteq \tau' + 1^\ell$ and therefore

$$\sigma' \trianglerighteq \tau' + 0^k 1^{\ell-k}.$$

Hence, from Lemma 5.5.10 we find

$$\sigma \trianglelefteq \tau \cup (0^k, \ell - k),$$

so we deduce that $(\sigma, k) \succeq (\tau \cup (0^k, \ell - k), k)$ using the dominance rule (Lemma 5.5.4). We now distinguish the two cases in Definition 5.5.9.

If $\tau \cup (0^k, \ell - k) = \tau +_i (\ell - k)$ for some $i$, then $\tau_i \leqslant k$ and we find

$$(\sigma, k) \succeq (\tau +_i (\ell - k), k) \succeq (\tau, k + (\ell - k)) = (\tau, \ell)$$

using the sum rule (Lemma 5.5.6).

If $\tau \cup (0^k, \ell - k) = \tau +_{i-1} (\ell - \tau_{i-1}) +_i (\tau_{i-1} - k)$ for some $i$, then again $\tau_i \leqslant k$ and we find

$$(\sigma, k) \succeq (\tau +_{i-1} (\ell - \tau_{i-1}) +_i (\tau_{i-1} - k), k) \succeq (\tau +_{i-1} (\ell - \tau_{i-1}), \tau_{i-1}) = (\tau, \ell)$$

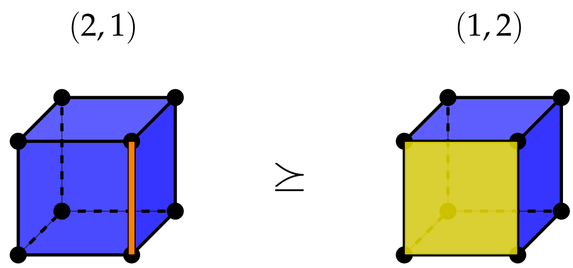
using the sum rule (Lemma 5.5.6) twice. $\square$

This theorem gives an easy criterion to compare parabolic transitivity types in the group $W$. It is a generalisation of the Livingstone–Wagner theorem (see Theorem 4.3.3) from $S_n$ to the group $W$. Moreover, in the case $G = C_1$, the parabolic types become the double partitions $(\sigma, \varnothing)$, so we have $k = 0$ and Theorem 5.5.11 specialises to Theorem 5 in [46], the generalisation of the Livingstone–Wagner theorem by Martin and Sagan for the symmetric group $S_n$.

Let us apply Theorem 5.5.11 to $W = C_2 \wr S_n$, the symmetry group of the $n$-dimensional cube.

**Example 5.5.12.** Let $W = C_2 \wr S_n$ be the hyperoctahedral group, also denoted as $B_n$. Then all types $\underline{\sigma}$ are simple.

First, let $n = 3$ and consider the parabolic types $\underline{\sigma} = (2,1)$ and $\underline{\tau} = (1,2)$. They correspond to the subgroups $H_{\underline{\sigma}} = S_2 \times B_1$ and $H_{\underline{\tau}} = S_1 \times B_2$. Geometrically, we have seen in Figure 5.2 that the double partition $(2,1)$ stands for edge-transitivity and the double partition $(1,2)$ stands for face-transitivity. Applying Theorem 5.5.11 gives $(2,1) \succeq (1,2)$, that is every $(2,1)$-transitive set is also $(1,2)$-transitive. Geometrically, this means that every subset of $C_2 \wr S_3$ that is transitive on the edges of a 3-dimensional cube is also transitive on the faces of a 3-dimensional cube.




We can generalise this observation to the $n$-dimensional cube. The stabiliser of a $t$-dimensional face of the cube is $S_{n-t} \times B_t$. Thus, the double partition $(n-t, t)$ describes transitivity on the $t$-dimensional faces of the $n$-dimensional cube. Theorem 5.5.11 gives

$$(n-t, t) \succeq (n-s, s) \iff t \leqslant s \text{ and } ((n-t) \cup (t)) \trianglelefteq ((n-s) \cup (s)).$$

We distinguish the cases $t \geqslant n-t$ and $t < n-t$.

If $t \geqslant n-t$, then Theorem 5.5.11 gives $(n-t, t) \succeq (n-s, s)$ for every $s \geqslant t$ and $(n-t, t) \not\succeq (n-s, s)$ for every $s < t$. Thus, if $n/2 \leqslant t \leqslant n-1$, then a set that acts transitively on the $t$-dimensional faces of an $n$-dimensional cube also acts transitively on the $(t+1)$-dimensional faces.

If $t < n-t$, then Theorem 5.5.11 gives $(n-t, t) \succeq (t, n-t)$ and $(n-t, t) \not\succeq (n-s, s)$ for every $s$ with $0 \leqslant s < n-t$ except $s = t$. Thus, if $0 \leqslant t < n/2$, then transitivity on the $t$-dimensional faces of an $n$-dimensional cube implies transitivity on the $(n-t)$-dimensional faces and we are in the first case again.

We summarise our findings in the following theorem.

**Theorem 5.5.13.** *Let $Y \subseteq C_2 \wr S_n$ be a non-empty subset.*

(a) *If $t$ is an integer with $n/2 \leqslant t < n$ and $Y$ is transitive on the faces of dimension $t$ of an $n$-dimensional cube, then $Y$ is transitive on the faces of dimension $t+1$.*

(b) *If $t$ is an integer with $1 \leqslant t < n/2$ and $Y$ is transitive on the faces of dimension $t$ of an $n$-dimensional cube, then $Y$ is transitive on the faces of dimension $n-t$.*

Similar results hold for complex regular polytopes, that is the cases $W = C_r \wr S_n$ with $r > 2$ in Theorem 5.5.11.

We strongly believe that the dominance rule and the sum rule characterise the relation $\succeq$ completely. That is, we make the following conjecture.

**Conjecture 5.5.14.** *Let $(\sigma, \rho)$ and $(\tau, \pi)$ be two double partitions of the same size. Then the following are equivalent.*

(a) $(\sigma, \rho) \succeq (\tau, \pi)$

(b) *There exists a sequence $(\sigma_i, \rho_i)$ with*

$$(\sigma, \rho) = (\sigma_1, \rho_1) \succeq (\sigma_2, \rho_2) \succeq \ldots \succeq (\sigma_k, \rho_k) = (\tau, \pi)$$

*such that every step $(\sigma_i, \rho_i) \succeq (\sigma_{i+1}, \rho_{i+1})$ for $i = 1, \ldots, k-1$ follows from Lemma 5.5.4 (dominance rule) or Lemma 5.5.6 (sum rule).*

We have verified this conjecture for the hyperoctahedral group $C_2 \wr S_n$ for every $n \leqslant 25$ for every double partition of size $n$. Note that the implication $(b) \implies (a)$ follows because the relation $\succeq$ is transitive.

We also have computational evidence that the following criterion characterises the relation $\succeq$.

**Conjecture 5.5.15.** *Let $(\sigma, \rho)$ and $(\tau, \pi)$ be two double partitions of the same size. Then the following are equivalent.*

(a) $(\sigma, \rho) \succeq (\tau, \pi)$

(b) *There exists a partition $\widetilde{\rho}$ with $|\widetilde{\rho}| = |\rho|$ and $\widetilde{\rho} \trianglerighteq \rho$ such that*

$$\widetilde{\rho} \subseteq \pi \text{ and } \sigma \cup \widetilde{\rho} \trianglelefteq \tau \cup \pi.$$

Notice that the condition in (b) is similar to the necessary conditions of Proposition 5.5.2. If $(\sigma, \rho) \succeq (\tau, \pi)$, then it follows that $\rho \trianglelefteq \pi$, which is equivalent to the existence of a partition $\widetilde{\rho}$ with $|\widetilde{\rho}| = |\rho|$ and $\widetilde{\rho} \trianglerighteq \rho$ such that $\widetilde{\rho} \subseteq \pi$. The conjecture claims that there is such a partition $\widetilde{\rho}$ that satisfies $\sigma \cup \widetilde{\rho} \trianglelefteq \tau \cup \pi$ while Proposition 5.5.2 only asserts that $\sigma \cup \rho \trianglelefteq \tau \cup \pi$. Notice that the stronger condition rules out the counterexample in Example 5.5.3 that shows that the conditions of Proposition 5.5.2 are not sufficient. Furthermore, notice that Theorem 5.5.11 is a special case of Conjecture 5.5.15 because $k \leqslant \ell \iff (k) \subseteq (\ell)$.

There is a relation between Conjecture 5.5.14 and Conjecture 5.5.15. We can think of the condition in Conjecture 5.5.15 as first using the dominance rule for the double partitions $(\sigma, \rho)$ and $(\sigma, \widetilde{\rho})$ and then using the sum rule. It is not hard to show that if $(\sigma, \widetilde{\rho}) \succeq (\tau, \pi)$ follows from a sequence of applications of the sum rule, then the conditions of Conjecture 5.5.15 are satisfied. Furthermore, through careful case analysis, it can be shown that a chain

$$(\sigma, \rho) = (\sigma_1, \rho_1) \succeq (\sigma_2, \rho_2) \succeq \ldots \succeq (\sigma_k, \rho_k) = (\tau, \pi)$$

where each step can be explained through the dominance rule and the sum rule can be transformed into a chain where the first steps follow from the dominance rule and the remaining steps follow from the sum rule. Hence, we get a condition very similar to the condition in Conjecture 5.5.15. We expect that Definition 5.5.9 can be generalised to skew tableaux, giving a way to prove the conjecture.

**Remark 5.5.16.** The results of this section can be generalised to non-simple types $\underline{\sigma} \in \Sigma_n(G)$. The partitions in $\underline{\sigma}$ are indexed by the subgroups of $G$. It is not hard to see that suitable generalisations of the dominance rule (Lemma 5.5.4) and the sum rule (Lemma 5.5.6) hold in this case.

First, it is clear that the dominance rule holds for every component $\underline{\sigma}(U)$ of $\underline{\sigma}$ independently as the proof of Lemma 5.5.4 works for a colouring coming from an arbitrary group $G$.

For the sum rule, consider two subgroups $U, V \leqslant G$ such that $U \subseteq V$ (equivalently $U^\circ \supseteq V^\circ$). Then, we can look at the double partition $(\underline{\sigma}(U), \underline{\sigma}(V))$ of two entries of $\underline{\sigma}$. For this double partition, the sum rule holds, that is, we can move some boxes from the partition $\underline{\sigma}(U)$ to the partition $\underline{\sigma}(V)$. The reason is that the condition $U \subseteq V$ ensures that the colouring can be copied as in the proof of Lemma 5.5.6. Thus, the sum rule holds for any two entries of $\underline{\sigma}$ indexed by $U$ and $V$ provided that $U \subseteq V$. It follows that the union rule and the exchange rule also hold because they are special cases of the sum rule.

Finally, the property from Proposition 5.5.2 (c) generalises to non-simple types by taking the union of all partitions $\underline{\sigma}(U)$ for $U \leqslant G$.

## 5.6 DESIGNS, CODES AND ORTHOGONAL POLYNOMIALS

Certain association schemes, namely $P$- and $Q$-polynomial schemes, are closely related to orthogonal polynomials in the sense that their character tables arise as evaluations of such polynomials (see [5] or [19]). The conjugacy class scheme of $C_r \wr S_n$ does not have these properties. Nevertheless, there is still a relationship to certain orthogonal polynomials, namely the Charlier polynomials. We show that there are characters associated with the Charlier polynomials which have a nice decomposition into irreducible characters. These characters and their decomposition can then be used to characterise codes and designs in the group $C_r \wr S_n$.

### 5.6.1 *Some Useful Results*

We first collect some results that will be useful in this section.

One useful technique is the well-known *binomial transform*. First, for $n, m \in \mathbb{N}_0$ with $n \geqslant m$, consider the well-known identity

$$\sum_{k=m}^{n} (-1)^{n-k} \binom{n}{k} \binom{k}{m} = \sum_{k=m}^{n} (-1)^{k-m} \binom{n}{k} \binom{k}{m} = \delta_{mn} \tag{5.9}$$

which can be obtained from the binomial theorem as

$$\begin{aligned}
\sum_{k=m}^{n} (-1)^{n-k} \binom{n}{k} \binom{k}{m} &= \sum_{k=m}^{n} (-1)^{n-k} \binom{n}{m} \binom{n-m}{k-m} \\
&= \binom{n}{m} \sum_{k=m}^{n} (-1)^{n-k} \binom{n-m}{k-m} \\
&= \binom{n}{m} \sum_{k=0}^{n-m} (-1)^{n-m-k} \binom{n-m}{k} \\
&= \binom{n}{m} (1 + (-1))^{n-m} = \delta_{nm}.
\end{aligned}$$

For a sequence $(a_k)_{k \in \mathbb{N}_0}$, we can consider the sequence $(s_n)_{n \in \mathbb{N}_0}$ defined by

$$s_n = \sum_{k=0}^{n} \binom{n}{k} a_k,$$

called the *binomial transform* of $(a_k)_{k \in \mathbb{N}_0}$. Now, multiplying this equation by $(-1)^{m-n} \binom{m}{n}$ and summing over $n$, we can use Equation (5.9) to find that

$$a_k = \sum_{n=0}^{k} (-1)^{k-n} \binom{k}{n} s_n \tag{5.10}$$

for every $k \in \mathbb{N}_0$. Switching from one sequence to the other is also called *binomial inversion*. Furthermore, if $t \in \mathbb{N}_0$ is an integer and we consider the sequence $(s_n)_{n=0,\dots,t}$ defined by

$$s_n = \sum_{k=0}^{t} \binom{k}{n} a_k,$$

then using similar calculations, Equation (5.9) implies that

$$a_k = \sum_{n=0}^{t} (-1)^{n-k} \binom{n}{k} s_n \tag{5.11}$$

for every $k \in \{0, \dots, t\}$.

We also make use of the *branching rule*. The group $S_{n-1}$ can be embedded into the group $S_n$ in a natural way (take the set of all permutations $\sigma \in S_n$ that fix $n$). Similarly, the group $C_r \wr S_{n-1}$ can be embedded into the group $C_r \wr S_n$. Hence, it is natural to ask what the restriction of an irreducible character of $C_r \wr S_n$ to $C_r \wr S_{n-1}$ looks like. This question is answered by the branching rule. Once we understand the restriction, the opposite question of inducing a character from $C_r \wr S_{n-1}$ to $C_r \wr S_n$ follows immediately from Frobenius reciprocity.

We first give the branching rule for the symmetric group and then the statement for the generalised symmetric group. First, we need a definition.

**Definition 5.6.1.** Let $n \in \mathbb{N}$ and $\lambda \vdash n$. The set $\lambda^-$ is the set of all partitions of $n-1$ that can be obtained from $\lambda$ by decreasing a part by 1. Similarly, for $n \in \mathbb{N}_0$ and $\lambda \vdash n$, the set $\lambda^+$ is the set of all partitions of $n+1$ that can be obtained from $\lambda$ by increasing a part by 1 or by adding a part of size 1 to $\lambda$.

Notice that we are only considering partitions, not compositions. Hence, we are ignoring all sequences obtained by increasing or decreasing a part that are not partitions. In other words, if $\lambda$ has multiple parts of the same size, then we may only increase the first of them and we may only decrease the last of them.

The sets $\lambda^-$ and $\lambda^+$ are illustrated in Figure 5.8 using the partition $(43311)$. The blue boxes are the boxes that can be removed to obtain a partition in $(43311)^-$ and the red boxes are the boxes that can be added to obtain a partition in $(43311)^+$.

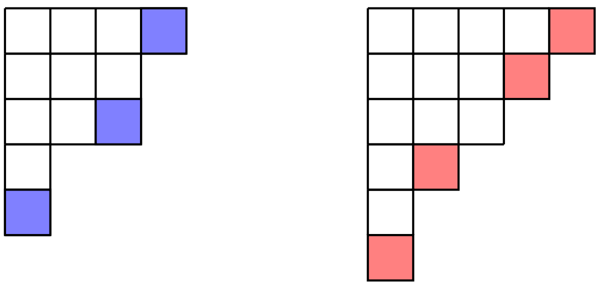


Figure 5.8: The sets $\lambda^-$ and $\lambda^+$

So we see that

$$(43311)^- = \{(33311), (43211), (4331)\},$$
$$(43311)^+ = \{(53311), (44311), (43321), (433111)\}.$$

The branching rule can now be stated in terms of the sets $\lambda^-$ and $\lambda^+$.

**Theorem 5.6.2.** *Let $n \in \mathbb{N}$ and $\lambda \vdash n$. Then we have*

(a) $\chi^{\lambda} \downarrow^{S_n}_{S_{n-1}} = \sum_{\mu \in \lambda^-} \chi^{\mu}$,

(b) $\chi^{\lambda} \uparrow^{S_{n+1}}_{S_n} = \sum_{\mu \in \lambda^+} \chi^{\mu}$.

This formula is well-known and gives rise to much of the combinatorial structure of the representation theory of the symmetric group. For a proof of the formula, see [54, Theorem 2.8.3] for example.

There is a variant of the branching rule for wreath products of the form $G \wr S_n$ (see [51, Theorem 10]). We adapt the statement to groups of the form $C_r \wr S_n$. First, we give the appropriate analogues of the sets $\lambda^-$ and $\lambda^+$ for partition tuples.

**Definition 5.6.3.** Let $n \in \mathbb{N}$ and $\underline{\lambda} \in \Lambda_n(C_r)$. The set $\underline{\lambda}^-$ is the set of all partition tuples $\underline{\mu} \in \Lambda_{n-1}(C_r)$ that can be obtained from $\underline{\lambda}$ by decreasing a part of a partition $\underline{\lambda}(g)$ for some $g \in C_r$ by 1. Similarly, for $n \in \mathbb{N}_0$ and $\underline{\lambda} \in \Lambda_n(C_r)$, the set $\underline{\lambda}^+$ is the set of all partition tuples $\underline{\mu} \in \Lambda_{n+1}(C_r)$ that can be obtained from $\underline{\lambda}$ by increasing a part of a partition $\underline{\lambda}(g)$ by 1 or by adding a part of size 1 to $\underline{\lambda}(g)$ for some $g \in C_r$.

In other words, $\underline{\lambda}^-$ (resp. $\underline{\lambda}^+$) is the set of all partition tuples that can be obtained from $\underline{\lambda}$ by replacing precisely one entry $\underline{\lambda}(g)$ with an element of $\underline{\lambda}(g)^-$ (resp. $\underline{\lambda}(g)^+$).

As an example, let us compute the sets $\underline{\lambda}^-$ and $\underline{\lambda}^+$ for an element of $\Lambda_n(C_r)$. We identify the elements of $C_r$ with the set $\{0, 1, \ldots, r-1\}$ and write an element $\underline{\lambda} \in \Lambda_n(C_r)$ as a tuple $\underline{\lambda} = \big(\underline{\lambda}(0), \underline{\lambda}(1), \ldots, \underline{\lambda}(r-1)\big)$ indexed by $0, \ldots, r-1$. When no confusion can arise, we omit the brackets of the partitions $\underline{\lambda}(g)$ for better readability. For example, we write $(2, 31, 1)$ instead of $((2), (31), (1))$.

For the partition tuple $(2, \varnothing, 1) \in \Lambda_3(C_3)$, we have

$$\begin{aligned}
(2, \varnothing, 1)^- &= \{(1, \varnothing, 1), (2, \varnothing, \varnothing)\}, \\
(2, \varnothing, 1)^+ &= \{(3, \varnothing, 1), (21, \varnothing, 1), (2, 1, 1), \\
&\qquad (2, \varnothing, 2), (2, \varnothing, 11)\}.
\end{aligned}$$

We can now state the branching rule for the group $C_r \wr S_n$.

**Theorem 5.6.4.** *Let $r, n \in \mathbb{N}$ and $\underline{\lambda} \in \Lambda_n(C_r)$. Then we have*

(a) $\chi^{\underline{\lambda}} \downarrow^{C_r \wr S_n}_{C_r \wr S_{n-1}} = \sum_{\underline{\mu} \in \underline{\lambda}^-} \chi^{\underline{\mu}}$,

(b) $\chi^{\underline{\lambda}} \uparrow^{C_r \wr S_{n+1}}_{C_r \wr S_n} = \sum_{\underline{\mu} \in \underline{\lambda}^+} \chi^{\underline{\mu}}$.

For a proof, see [51, Theorem 10]. We remark that a proof can also be obtained from our proof of Theorem 5.3.10 by going through the proof for the case $H = C_r \wr S_{n-1}$ in more detail. Further, note that Theorem 5.6.4 reduces to Theorem 5.6.2 in the case $r = 1$.

A well-known result that will be useful together with the branching rule is the following. It connects induction and restriction of characters.

**Theorem 5.6.5.** *Let $G$ be a finite group and let $H$ be a subgroup of $G$. Furthermore, let $\chi$ be a character of $H$ and let $\psi$ be a character of $G$. Then we have*

$$\left(\chi \uparrow_H^G\right) \cdot \psi = \left(\chi \cdot \psi \downarrow_H^G\right) \uparrow_H^G .$$

For a proof, see [57, Chapter 7.2].

### 5.6.2 *Charlier Polynomials*

The *Charlier polynomial* of degree $k$ with parameter $a \in \mathbb{R}$, $a > 0$, is given by

$$C_k^{(a)}(x) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} a^{-j} (x)_j,$$

where $(x)_j = x(x-1)\cdots(x-j+1)$ is the falling factorial. Some properties of these polynomials can be found in [36, Chapter 1.12]. The first polynomials are

$$\begin{aligned} C_0^{(a)}(x) &= 1, \\ C_1^{(a)}(x) &= a^{-1}x - 1, \\ C_2^{(a)}(x) &= a^{-2}x^2 - (2a^{-1} + a^{-2})x + 1. \end{aligned}$$

In fact, $C_1^{(a)}, C_2^{(a)}, \ldots$ form a system of orthogonal polynomials with respect to the Poisson distribution with mean $a$. That is, we have

$$\sum_{i=0}^{\infty} \frac{a^i}{i!} C_k^{(a)}(i) C_\ell^{(a)}(i) = 0 \quad \text{for } k \neq \ell. \tag{5.12}$$

Sometimes the Charlier polynomials are defined to be the polynomials $a^k C_k^{(a)}$ or $(-1)^k C_k^{(a)}$. However, all of them satisfy the orthogonality relation (5.12).

It is well-known that the monic Charlier polynomials $p_k^{(a)}$ satisfy the recurrence

$$x p_k(x) = p_{k+1}(x) + (k+a) p_k(x) + k a p_{k-1}(x)$$

for every $k \in \mathbb{N}$ where $p_0^{(a)}(x) = 1$ and $p_1^{(a)}(x) = x - a$. Since the leading coefficient of $C_k^{(a)}$ is $a^{-k}$, it follows that the Charlier polynomials $C_k^{(a)}$ in the way we defined them satisfy the recurrence

$$xC_k^{(a)}(x) = aC_{k+1}^{(a)}(x) + (k+a)C_k^{(a)}(x) + kC_{k-1}^{(a)}(x) \tag{5.13}$$

for every $k \in \mathbb{N}$.

For $g \in C_r \wr S_n$, let

$$\theta(g) = \frac{1}{r} \left|\{(c,i) \in [r] \times [n] \mid g(c,i) = (c,i)\}\right|.$$

That is, $\theta(g)$ is the number of fixed points of the natural action of $C_r \wr S_n$ on $[r] \times [n]$, scaled by $1/r$. Then $\theta$ is a class function of $C_r \wr S_n$ taking values in $\{0, 1, \ldots, n\}$ (note that $g(c,i) = (c,i)$ for some $c \in [r]$ and $i \in [n]$ implies $g(\widetilde{c}, i) = (\widetilde{c}, i)$ for every $\widetilde{c} \in [r]$). We write $w_i$ for the number of elements $g \in C_r \wr S_n$ satisfying $\theta(g) = i$. Using the inclusion-exclusion principle, one can derive the expression

$$w_i = \frac{n!\, r^{n-i}}{i!} \sum_{j=0}^{n-i} \frac{(-1)^j}{j!\, r^j},$$

see for example [14, Section 7]. We shall later see that this expression also follows from our results (see Remark 5.6.14).

The function $\theta$ is a class function and it defines a uniformly distributed discrete random variable on $C_r \wr S_n$. It was shown in [14, Prop. 7.4] that, for $k \leqslant n$, the $k$-th moment of $\theta$ equals the $k$-th moment of the Poisson distribution with mean $1/r$ (the condition $k \leqslant n$ was erroneously omitted in [14, Prop. 7.4]). The $k$-th moment of the Poisson distribution is given by the expression

$$e^{-\frac{1}{r}} \sum_{i=0}^{\infty} \frac{\left(\frac{1}{r}\right)^i}{i!} \cdot i^k.$$

Hence, we obtain

$$\mathbb{E}[\theta^k] = \sum_{g \in G} \frac{1}{|G|} \cdot \theta(g)^k = \frac{1}{|G|} \sum_{i=0}^{n} w_i \cdot i^k = e^{-\frac{1}{r}} \sum_{i=0}^{\infty} \frac{\left(\frac{1}{r}\right)^i}{i!} \cdot i^k \tag{5.14}$$

for every $k \leqslant n$. We mention in passing that the $k$-th moment of the Poisson distribution with mean $1/r$ is given by

$$\sum_{i=1}^{k} \left(\frac{1}{r}\right)^i S(k,i),$$

where $S(k,i)$ are the Stirling numbers of the second kind, counting the number of partitions of a set of size $k$ into $i$ non-empty subsets.

As $C_k^{(1/r)}C_l^{(1/r)}$ is a polynomial of degree $k+l$, we find from (5.12) and (5.14) that the Charlier polynomials also satisfy the orthogonality relation

$$\sum_{i=0}^{n} w_i\, C_k^{(1/r)}(i)C_\ell^{(1/r)}(i) = 0 \quad \text{for } k \neq \ell \text{ and } k+\ell \leqslant n. \tag{5.15}$$

With every polynomial $f(x) = f_n x^n + \cdots + f_1 x + f_0$ in $\mathbb{R}[x]$, we associate the class function $f(\theta) = f_n\theta^n + \cdots + f_1\theta + f_0 1_{C_r \wr S_n}$. This induces an algebra homomorphism from $\mathbb{R}[x]$ to the set of class functions of $C_r \wr S_n$.

Let $\xi_j$ be the permutation character on ordered $j$-tuples of pairs $(c_k, i_k) \in [r]\times[n]$ with all $i_k$ pairwise distinct. By convention, $\xi_0$ is the trivial character of $C_r \wr S_n$. Note that

$$\xi_j = r^j\theta(\theta-1)\ldots(\theta-j+1) = r^j\,(\theta)_j.$$

Hence, we have

$$C_k^{(1/r)}(\theta) = \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\xi_j \quad \text{for } k = 0,1,\ldots,n, \tag{5.16}$$

and from (5.10), we obtain

$$\xi_j = \sum_{k=0}^{j}\binom{j}{k}C_k^{(1/r)}(\theta) \quad \text{for } j = 0,1,\ldots,n. \tag{5.17}$$

For $0 \leqslant k \leqslant n/2$, we now determine the irreducible constituents of $C_k^{(1/r)}(\theta)$. Recall that for two elements $\underline{\lambda} \in \Lambda_\ell(G)$ and $\underline{\mu} \in \Lambda_m(G)$, their union $\underline{\lambda} \cup \underline{\mu} \in \Lambda_{\ell+m}(G)$ is defined as $(\underline{\lambda} \cup \underline{\mu})(g) = \underline{\lambda}(g) \cup \underline{\mu}(g)$ for every $g \in G$.

**Theorem 5.6.6.** *For each k satisfying* $0 \leqslant k \leqslant n/2$, *we have*

$$C_k^{(1/r)}(\theta) = \sum_{\underline{\lambda}\in\Lambda_k(C_r)} \deg(\chi^{\underline{\lambda}})\,\chi^{(n-k,\varnothing,\ldots,\varnothing)\cup\underline{\lambda}}.$$

*In particular,* $C_k^{(1/r)}(\theta)$ *is a character of* $C_r \wr S_n$, *and for* $\underline{\lambda} \in \Lambda_n(C_r)$ *we have*

$$\left\langle C_k^{(1/r)}(\theta), \chi^{\underline{\lambda}}\right\rangle \neq 0 \iff \underline{\lambda}(0)_1 = n-k.$$

*Proof.* From (5.13), we find that for $a = 1/r$ we have the recurrence

$$rxC_k^{(1/r)}(x) = C_{k+1}^{(1/r)}(x) + (rk+1)C_k^{(1/r)}(x) + rkC_{k-1}^{(1/r)}(x)$$

for every $k \in \mathbb{N}$. The Charlier polynomials are uniquely defined by this recurrence and the first two polynomials $C_0^{(1/r)}(x) = 1$ and $C_1^{(1/r)}(x) = rx - 1$. Letting $x = \theta$, we find the recurrence

$$r\theta C_k^{(1/r)}(\theta) = C_{k+1}^{(1/r)}(\theta) + (rk+1)C_k^{(1/r)}(\theta) + rkC_{k-1}^{(1/r)}(\theta).$$

for every $k \in \mathbb{N}$. This is an identity of class functions.

Let

$$F_k^{(r)} = \sum_{\underline{\lambda} \in \Lambda_k(C_r)} \deg(\chi^{\underline{\lambda}})\, \chi^{(n-k,\varnothing,\ldots,\varnothing)\cup\underline{\lambda}}$$

denote the proposed expression for $C_k^{(1/r)}(\theta)$. We show $C_k^{(1/r)}(\theta) = F_k^{(r)}$ for $k \leqslant n/2$ by showing that $C_0^{(1/r)}(\theta) = F_0^{(r)}$ and $C_1^{(1/r)}(\theta) = F_1^{(r)}$ and then showing that $F_k^{(r)}$ satisfies the recurrence

$$r\theta F_k^{(r)} = F_{k+1}^{(r)}(\theta) + (rk+1)F_k^{(r)} + rkF_{k-1}^{(r)} \tag{5.18}$$

for every $k \in \mathbb{N}$ with $1 \leqslant k \leqslant (n-2)/2$.

First, notice that

$$\begin{aligned} C_0^{(1/r)}(\theta) &= 1_{C_r \wr S_n} = \chi^{(n,\varnothing,\ldots,\varnothing)} = F_0^{(r)}, \\ C_1^{(1/r)}(\theta) &= r\theta - 1_{C_r \wr S_n} = \xi_1 - \chi^{(n,\varnothing,\ldots,\varnothing)}. \end{aligned}$$

Notice that $\xi_1$ is the fixed point character of $C_r \wr S_n$ in its natural action. The stabiliser of a point is $C_r \wr S_{n-1}$, so $\xi_1 = 1 \uparrow_{C_r \wr S_{n-1}}^{C_r \wr S_n}$. The trivial character of $C_r \wr S_{n-1}$ is indexed by the partition tuple $(n-1, \varnothing, \ldots, \varnothing)$. Hence, we find from the branching rule (Theorem 5.6.4) that

$$\xi_1 = 1 \uparrow_{C_r \wr S_{n-1}}^{C_r \wr S_n} = \sum_{\underline{\mu} \in \underline{\lambda}^+} \chi^{\underline{\mu}}$$

where $\underline{\lambda} = (n-1, \varnothing, \ldots, \varnothing)$. The partition tuples in the set $(n-1, \varnothing, \ldots, \varnothing)^+$ are easily seen to be

$$\begin{aligned} (n-1, \varnothing, \ldots, \varnothing)^+ = &\{(n, \varnothing, \ldots, \varnothing), ((n-1,1), \varnothing, \ldots, \varnothing), \\ &(n-1, 1, \varnothing, \ldots, \varnothing), \ldots, (n-1, \varnothing, \ldots, \varnothing, 1). \end{aligned}$$

These are precisely the partition tuples of the form $(n-1, \varnothing, \ldots, \varnothing) \cup \underline{\mu}$ where $\underline{\mu} \in \Lambda_1(C_r)$ together with the partition tuple $(n, \varnothing, \ldots, \varnothing)$ which indexes the trivial character. As $C_r \wr S_1 \cong C_r$, we find that all characters $\chi^{\underline{\mu}}$ with $\underline{\mu} \in \Lambda_1(C_r)$ are 1-dimensional. Hence, we find that

$$C_1^{(1/r)}(\theta) = \xi_1 - \chi^{(n,\varnothing,\ldots,\varnothing)} = \sum_{\underline{\mu} \in \Lambda_1(C_r)} \deg(\chi^{\underline{\mu}})\, \chi^{(n-1,\varnothing,\ldots,\varnothing)\cup\underline{\mu}} = F_1^{(r)}.$$

Now let $k \in \mathbb{N}$ with $1 \leqslant k \leqslant (n-2)/2$. Since $r\theta = \xi_1 = 1 \uparrow_{C_r \wr S_{n-1}}^{C_r \wr S_n}$, we first investigate the product $\xi_1 \chi^{(n-k,\varnothing,\ldots,\varnothing)\cup\underline{\lambda}}$ for some $\underline{\lambda} \in \Lambda_k(C_r)$. From Theorem 5.6.5, we find

$$\begin{aligned}
\xi_1 \chi^{(n-k,\varnothing,\ldots,\varnothing)\cup\underline{\lambda}} &= \left(1 \uparrow_{C_r \wr S_{n-1}}^{C_r \wr S_n}\right) \chi^{(n-k,\varnothing,\ldots,\varnothing)\cup\underline{\lambda}} \\
&= \left(1 \cdot \chi^{(n-k,\varnothing,\ldots,\varnothing)\cup\underline{\lambda}} \downarrow_{C_r \wr S_{n-1}}^{C_r \wr S_n}\right) \uparrow_{C_r \wr S_{n-1}}^{C_r \wr S_n} \\
&= \left(\chi^{(n-k,\varnothing,\ldots,\varnothing)\cup\underline{\lambda}} \downarrow_{C_r \wr S_{n-1}}^{C_r \wr S_n}\right) \uparrow_{C_r \wr S_{n-1}}^{C_r \wr S_n} .
\end{aligned}$$

The restriction of the character $\chi^{(n-k,\varnothing,\ldots,\varnothing)\cup\underline{\lambda}}$ to $C_r \wr S_{n-1}$ can be computed using the branching rule (Theorem 5.6.4). Notice that because we assume $k \leqslant (n-2)/2$, it follows that $k < n-k$. Hence, the partition tuple $\underline{\lambda}$ cannot contain a part of size $n-k$. It follows that when using the branching rule for the partition tuple $(n-k,\varnothing,\ldots,\varnothing) \cup \underline{\lambda}$, we may decrease the part of size $n-k$ and the parts of $\underline{\lambda}$ independently. Thus, we obtain

$$\chi^{(n-k,\varnothing,\ldots,\varnothing)\cup\underline{\lambda}} \downarrow_{C_r \wr S_{n-1}}^{C_r \wr S_n} = \chi^{(n-k-1,\varnothing,\ldots,\varnothing)\cup\underline{\lambda}} + \sum_{\underline{\mu} \in \underline{\lambda}^-} \chi^{(n-k,\varnothing,\ldots,\varnothing)\cup\underline{\mu}}.$$

Similarly, we can compute the induction of the characters using the branching rule. Notice that we have $k < n-k-1$. It follows that when using the branching rule for the partition tuple $(n-k-1,\varnothing,\ldots,\varnothing) \cup \underline{\lambda}$, we may increase the part of size $n-k-1$ and the parts of $\underline{\lambda}$ independently. Thus, we obtain

$$\chi^{(n-k-1,\varnothing,\ldots,\varnothing)\cup\underline{\lambda}} \uparrow_{C_r \wr S_{n-1}}^{C_r \wr S_n} = \chi^{(n-k,\varnothing,\ldots,\varnothing)\cup\underline{\lambda}} + \sum_{\underline{\mu} \in \underline{\lambda}^+} \chi^{(n-k-1,\varnothing,\ldots,\varnothing)\cup\underline{\mu}}$$

and

$$\begin{aligned}
&\left(\sum_{\underline{\mu} \in \underline{\lambda}^-} \chi^{(n-k,\varnothing,\ldots,\varnothing)\cup\underline{\mu}}\right) \uparrow_{C_r \wr S_{n-1}}^{C_r \wr S_n} \\
&= \sum_{\underline{\mu} \in \underline{\lambda}^-} \left(\chi^{(n-k,\varnothing,\ldots,\varnothing)\cup\underline{\mu}}\right) \uparrow_{C_r \wr S_{n-1}}^{C_r \wr S_n} \\
&= \sum_{\underline{\mu} \in \underline{\lambda}^-} \chi^{(n-k+1,\varnothing,\ldots,\varnothing)\cup\underline{\mu}} + \sum_{\underline{\mu} \in \underline{\lambda}^-} \sum_{\underline{\tau} \in \underline{\mu}^+} \chi^{(n-k,\varnothing,\ldots,\varnothing)\cup\underline{\tau}}.
\end{aligned}$$

Putting it all together, we find

$$\begin{aligned}
&\xi_1 \chi^{(n-k,\varnothing,\ldots,\varnothing)\cup\underline{\lambda}} \\
= &\left(\chi^{(n-k,\varnothing,\ldots,\varnothing)\cup\underline{\lambda}} \downarrow^{C_r\wr S_n}_{C_r\wr S_{n-1}}\right) \uparrow^{C_r\wr S_n}_{C_r\wr S_{n-1}} \\
= &\chi^{(n-k,\varnothing,\ldots,\varnothing)\cup\underline{\lambda}} \\
&+ \sum_{\underline{\mu}\in\underline{\lambda}^+} \chi^{(n-k-1,\varnothing,\ldots,\varnothing)\cup\underline{\mu}} \\
&+ \sum_{\underline{\mu}\in\underline{\lambda}^-} \chi^{(n-k+1,\varnothing,\ldots,\varnothing)\cup\underline{\mu}} \\
&+ \sum_{\underline{\mu}\in\underline{\lambda}^-} \sum_{\underline{\tau}\in\underline{\mu}^+} \chi^{(n-k,\varnothing,\ldots,\varnothing)\cup\underline{\tau}}.
\end{aligned}$$

Now we can compute the product $r\theta F_k^{(r)}$ and obtain

$$\begin{aligned}
&r\theta F_k^{(r)} = \xi_1 F_k^{(r)} = \sum_{\underline{\lambda}\in\Lambda_k(C_r)} \deg(\chi^{\underline{\lambda}})\, \xi_1 \chi^{(n-k,\varnothing,\ldots,\varnothing)\cup\underline{\lambda}} \\
= &\sum_{\underline{\lambda}\in\Lambda_k(C_r)} \deg(\chi^{\underline{\lambda}})\, \chi^{(n-k,\varnothing,\ldots,\varnothing)\cup\underline{\lambda}} \\
&+ \sum_{\underline{\lambda}\in\Lambda_k(C_r)} \deg(\chi^{\underline{\lambda}}) \sum_{\underline{\mu}\in\underline{\lambda}^+} \chi^{(n-k-1,\varnothing,\ldots,\varnothing)\cup\underline{\mu}} \\
&+ \sum_{\underline{\lambda}\in\Lambda_k(C_r)} \deg(\chi^{\underline{\lambda}}) \sum_{\underline{\mu}\in\underline{\lambda}^-} \chi^{(n-k+1,\varnothing,\ldots,\varnothing)\cup\underline{\mu}} \\
&+ \sum_{\underline{\lambda}\in\Lambda_k(C_r)} \deg(\chi^{\underline{\lambda}}) \sum_{\underline{\mu}\in\underline{\lambda}^-} \sum_{\underline{\tau}\in\underline{\mu}^+} \chi^{(n-k,\varnothing,\ldots,\varnothing)\cup\underline{\tau}} \\
= &F_k^{(r)} + A + B + C
\end{aligned}$$

where

$$\begin{aligned}
A &= \sum_{\underline{\lambda}\in\Lambda_k(C_r)} \deg(\chi^{\underline{\lambda}}) \sum_{\underline{\mu}\in\underline{\lambda}^+} \chi^{(n-k-1,\varnothing,\ldots,\varnothing)\cup\underline{\mu}}, \\
B &= \sum_{\underline{\lambda}\in\Lambda_k(C_r)} \deg(\chi^{\underline{\lambda}}) \sum_{\underline{\mu}\in\underline{\lambda}^-} \chi^{(n-k+1,\varnothing,\ldots,\varnothing)\cup\underline{\mu}}, \\
C &= \sum_{\underline{\lambda}\in\Lambda_k(C_r)} \deg(\chi^{\underline{\lambda}}) \sum_{\underline{\mu}\in\underline{\lambda}^-} \sum_{\underline{\tau}\in\underline{\mu}^+} \chi^{(n-k,\varnothing,\ldots,\varnothing)\cup\underline{\tau}}.
\end{aligned}$$

In view of Equation (5.18), it remains to show that $A = F_{k+1}^{(r)}$, $B = rkF_{k-1}^{(r)}$ and $C = rkF_k^{(r)}$.

First, notice that summing over all pairs $(\underline{\lambda}, \underline{\mu})$ where $\underline{\lambda} \in \Lambda_k(C_r)$ and $\underline{\mu} \in \underline{\lambda}^+$ is equivalent to summing over all pairs $(\underline{\lambda}, \underline{\mu})$ where $\underline{\mu} \in \Lambda_{k+1}(C_r)$ and $\underline{\lambda} \in \underline{\mu}^-$.

This is easily seen by double counting the set of all pairs $(\underline{\lambda}, \underline{\mu})$ such that $\underline{\lambda} \in \Lambda_k(C_r)$, $\underline{\mu} \in \Lambda_{k+1}(C_r)$, $\underline{\lambda}(g) \subseteq \underline{\mu}(g)$ for some $g \in C_r$ and $\underline{\lambda}(h) = \underline{\mu}(h)$ for all $h \neq g$. This condition ensures that $\underline{\mu} \in \underline{\lambda}^+$ which is equivalent to $\underline{\lambda} \in \underline{\mu}^-$. It follows that

$$\begin{aligned} A &= \sum_{\underline{\lambda} \in \Lambda_k(C_r)} \deg(\chi^{\underline{\lambda}}) \sum_{\underline{\mu} \in \underline{\lambda}^+} \chi^{(n-k-1,\varnothing,\dots,\varnothing)\cup\underline{\mu}} \\ &= \sum_{\underline{\lambda} \in \Lambda_k(C_r)} \sum_{\underline{\mu} \in \underline{\lambda}^+} \deg(\chi^{\underline{\lambda}})\, \chi^{(n-k-1,\varnothing,\dots,\varnothing)\cup\underline{\mu}} \\ &= \sum_{\underline{\mu} \in \Lambda_{k+1}(C_r)} \sum_{\underline{\lambda} \in \underline{\mu}^-} \deg(\chi^{\underline{\lambda}})\, \chi^{(n-k-1,\varnothing,\dots,\varnothing)\cup\underline{\mu}} \\ &= \sum_{\underline{\mu} \in \Lambda_{k+1}(C_r)} \left( \sum_{\underline{\lambda} \in \underline{\mu}^-} \deg(\chi^{\underline{\lambda}}) \right) \chi^{(n-k-1,\varnothing,\dots,\varnothing)\cup\underline{\mu}}. \end{aligned}$$

Now let $\underline{\mu} \in \Lambda_{k+1}(C_r)$ and recall that $\deg(\chi^{\underline{\mu}}) = \chi^{\underline{\mu}}(e)$. Since restricting a character does not change its dimension, we find from the branching rule that

$$\begin{aligned} \deg(\chi^{\underline{\mu}}) &= \deg\left(\chi^{\underline{\mu}} \downarrow^{C_r \wr S_{k+1}}_{C_r \wr S_k}\right) = \left(\chi^{\underline{\mu}} \downarrow^{C_r \wr S_{k+1}}_{C_r \wr S_k}\right)(e) \\ &= \sum_{\underline{\lambda} \in \underline{\mu}^-} \chi^{\underline{\lambda}}(e) = \sum_{\underline{\lambda} \in \underline{\mu}^-} \deg(\chi^{\underline{\lambda}}). \end{aligned}$$

Thus, we deduce that $A = F^{(r)}_{k+1}$.

Similarly, notice that summing over all pairs $(\underline{\lambda}, \underline{\mu})$ where $\underline{\lambda} \in \Lambda_k(C_r)$ and $\underline{\mu} \in \underline{\lambda}^-$ is equivalent to summing over all pairs $(\underline{\lambda}, \underline{\mu})$ where $\underline{\mu} \in \Lambda_{k-1}(C_r)$ and $\underline{\lambda} \in \underline{\mu}^+$. It follows that

$$\begin{aligned} B &= \sum_{\underline{\lambda} \in \Lambda_k(C_r)} \deg(\chi^{\underline{\lambda}}) \sum_{\underline{\mu} \in \underline{\lambda}^-} \chi^{(n-k+1,\varnothing,\dots,\varnothing)\cup\underline{\mu}} \\ &= \sum_{\underline{\lambda} \in \Lambda_k(C_r)} \sum_{\underline{\mu} \in \underline{\lambda}^-} \deg(\chi^{\underline{\lambda}})\, \chi^{(n-k+1,\varnothing,\dots,\varnothing)\cup\underline{\mu}} \\ &= \sum_{\underline{\mu} \in \Lambda_{k-1}(C_r)} \sum_{\underline{\lambda} \in \underline{\mu}^+} \deg(\chi^{\underline{\lambda}})\, \chi^{(n-k+1,\varnothing,\dots,\varnothing)\cup\underline{\mu}} \\ &= \sum_{\underline{\mu} \in \Lambda_{k-1}(C_r)} \left( \sum_{\underline{\lambda} \in \underline{\mu}^+} \deg(\chi^{\underline{\lambda}}) \right) \chi^{(n-k+1,\varnothing,\dots,\varnothing)\cup\underline{\mu}}. \end{aligned}$$

Now let $\underline{\mu} \in \Lambda_{k-1}(C_r)$. Since inducing a character from $C_r \wr S_{k-1}$ to $C_r \wr S_k$ changes its degree by $\frac{|C_r \wr S_k|}{|C_r \wr S_{k-1}|} = rk$, we find that

$$\begin{aligned} rk \deg(\chi^{\underline{\mu}}) &= \deg\left(\chi^{\underline{\mu}} \uparrow^{C_r \wr S_k}_{C_r \wr S_{k-1}}\right) = \left(\chi^{\underline{\mu}} \uparrow^{C_r \wr S_k}_{C_r \wr S_{k-1}}\right)(e) \\ &= \sum_{\underline{\lambda} \in \underline{\mu}^+} \chi^{\underline{\lambda}}(e) = \sum_{\underline{\lambda} \in \underline{\mu}^+} \deg(\chi^{\underline{\lambda}}). \end{aligned}$$

Thus, we deduce that $B = rkF^{(r)}_{k-1}$.

Finally, we turn to $C$. Notice that summing over all triples $(\underline{\lambda}, \underline{\mu}, \underline{\tau})$ with $\underline{\lambda} \in \Lambda_k(C_r)$ such that $\underline{\mu} \in \underline{\lambda}^-$ and $\underline{\tau} \in \underline{\mu}^+$ is equivalent to summing over all triples $(\underline{\lambda}, \underline{\mu}, \underline{\tau})$ with $\underline{\tau} \in \Lambda_k(C_r)$ such that $\underline{\mu} \in \underline{\tau}^-$ and $\underline{\lambda} \in \underline{\mu}^+$. It follows that

$$\begin{aligned} C &= \sum_{\underline{\lambda} \in \Lambda_k(C_r)} \deg(\chi^{\underline{\lambda}}) \sum_{\underline{\mu} \in \underline{\lambda}^-} \sum_{\underline{\tau} \in \underline{\mu}^+} \chi^{(n-k,\varnothing,\ldots,\varnothing)\cup\underline{\tau}} \\ &= \sum_{\underline{\lambda} \in \Lambda_k(C_r)} \sum_{\underline{\mu} \in \underline{\lambda}^-} \sum_{\underline{\tau} \in \underline{\mu}^+} \deg(\chi^{\underline{\lambda}})\, \chi^{(n-k,\varnothing,\ldots,\varnothing)\cup\underline{\tau}} \\ &= \sum_{\underline{\tau} \in \Lambda_k(C_r)} \sum_{\underline{\mu} \in \underline{\tau}^-} \sum_{\underline{\lambda} \in \underline{\mu}^+} \deg(\chi^{\underline{\lambda}})\, \chi^{(n-k,\varnothing,\ldots,\varnothing)\cup\underline{\tau}} \\ &= \sum_{\underline{\tau} \in \Lambda_k(C_r)} \left( \sum_{\underline{\mu} \in \underline{\tau}^-} \sum_{\underline{\lambda} \in \underline{\mu}^+} \deg(\chi^{\underline{\lambda}}) \right) \chi^{(n-k,\varnothing,\ldots,\varnothing)\cup\underline{\tau}}. \end{aligned}$$

Now let $\underline{\tau} \in \Lambda_k(C_r)$. As before, we find

$$\begin{aligned} rk\deg(\chi^{\underline{\tau}}) = \deg\left(\chi^{\underline{\tau}} \downarrow^{C_r \wr S_k}_{C_r \wr S_{k-1}} \uparrow^{C_r \wr S_k}_{C_r \wr S_{k-1}}\right) &= \left(\chi^{\underline{\tau}} \downarrow^{C_r \wr S_k}_{C_r \wr S_{k-1}} \uparrow^{C_r \wr S_k}_{C_r \wr S_{k-1}}\right)(e) \\ = \sum_{\underline{\mu} \in \underline{\tau}^-} \sum_{\underline{\lambda} \in \underline{\mu}^+} \chi^{\underline{\lambda}}(e) &= \sum_{\underline{\mu} \in \underline{\tau}^-} \sum_{\underline{\lambda} \in \underline{\mu}^+} \deg(\chi^{\underline{\lambda}}). \end{aligned}$$

Thus, we deduce that $C = rkF^{(r)}_k$ which completes the proof. □

In other words, Theorem 5.6.6 implies that for $0 \leqslant k \leqslant n/2$, the character $C^{(1/r)}_k(\theta)$ decomposes into those characters $\chi^{\underline{\lambda}}$ such that the biggest part of $\underline{\lambda}(0)$ is $n-k$. Furthermore, notice that the case $r = 1$ in Theorem 5.6.6 gives exactly the statement of Theorem 7 in [62].

We get the following corollary from Theorem 5.6.6.

**Corollary 5.6.7.** *For $0 \leqslant k, \ell \leqslant n/2$, we have*

$$\langle C^{(1/r)}_k(\theta), C^{(1/r)}_\ell(\theta) \rangle = \delta_{k\ell} r^k k!.$$

*Proof.* We find from Theorem 5.6.6 that

$$\begin{aligned} &\langle C^{(1/r)}_k(\theta), C^{(1/r)}_\ell(\theta) \rangle \\ &= \sum_{\underline{\lambda} \in \Lambda_k(C_r)} \sum_{\underline{\mu} \in \Lambda_\ell(C_r)} \deg(\chi^{\underline{\lambda}}) \deg(\chi^{\underline{\mu}}) \langle \chi^{(n-k,\varnothing,\ldots,\varnothing)\cup\underline{\lambda}}, \chi^{(n-\ell,\varnothing,\ldots,\varnothing)\cup\underline{\mu}} \rangle. \end{aligned}$$

Since $k, \ell \leqslant n/2$, we have that $n-k, n-\ell \geqslant n/2$ and thus $n-k$ and $n-\ell$ are the biggest parts of the partitions in the entry indexed by 0. Thus, the inner products can only be non-zero if $n-k = n-\ell$, equivalently $k=\ell$. In this case, we find that

$$\begin{aligned}
&\sum_{\underline{\lambda}\in\Lambda_k(C_r)} \sum_{\underline{\mu}\in\Lambda_k(C_r)} \deg(\chi^{\underline{\lambda}}) \deg(\chi^{\underline{\mu}}) \langle \chi^{(n-k,\varnothing,\ldots,\varnothing)\cup\underline{\lambda}}, \chi^{(n-k,\varnothing,\ldots,\varnothing)\cup\underline{\mu}} \rangle \\
&= \sum_{\underline{\lambda}\in\Lambda_k(C_r)} \sum_{\underline{\mu}\in\Lambda_k(C_r)} \deg(\chi^{\underline{\lambda}}) \deg(\chi^{\underline{\mu}})\, \delta_{\underline{\lambda}\,\underline{\mu}} \\
&= \sum_{\underline{\lambda}\in\Lambda_k(C_r)} \deg(\chi^{\underline{\lambda}})^2.
\end{aligned}$$

Now the formula

$$\sum_{\chi\in\mathrm{Irr}(G)} \deg(\chi)^2 = |G|$$

finishes the proof. □

**Remark 5.6.8.** Observe that $\xi_k$ is the induction of the trivial character of the subgroup $(S_1)^k \times (C_r \wr S_{n-k})$ to $C_r \wr S_n$. An application of Theorem 5.3.10 with $\underline{\sigma} \in \Sigma_n(C_r)$ given by $\underline{\sigma}(\{0\}) = (1^k)$ and $\underline{\sigma}(C_r \wr S_n) = (n-k)$ therefore gives

$$\langle \xi_k, \chi^{\underline{\lambda}} \rangle \neq 0 \iff ((1^k), n-k) \to \underline{\lambda}$$

for each $\underline{\lambda} \in \Lambda_n(C_r)$. From the definition of the relation $\to$, it is easy to see that the elements $\underline{\lambda}$ satisfying $((1^k), n-k) \to \underline{\lambda}$ are precisely those satisfying $\underline{\lambda}(0)_1 \geqslant n-k$.

It follows from Theorem 5.6.6 that for $0 \leqslant k \leqslant n/2$, inputting $\theta$ into the $k$-th Charlier polynomial $C_k^{(1/r)}$ gives the irreducible characters $\chi^{\underline{\lambda}}$ that satisfy $\underline{\lambda}(0)_1 = n-k$. These are precisely the irreducible constituents $\chi^{\underline{\lambda}}$ appearing in the decomposition of $\xi_k$ but not in the decomposition of $\xi_{k-1}$.

### 5.6.3 *Designs and Codes*

Henceforth, we call a $((1^t), n-t)$-transitive set $Y$ of $C_r \wr S_n$ a *t-design*. In the symmetric group, that is the case $r=1$, a $t$-design is simply a $t$-transitive subset. The *index* of $Y$ is the number of elements in $Y$ that stabilise an arbitrary $((1^t), n-t)$-tabloid (so the index is the constant $c$ in the definition of a transitive set). Thus, a $t$-design is transitive on the $t$-tuples $((c_1, i_1), \ldots, (c_t, i_t))$ where the elements $c_k \in [r]$ are arbitrary and the elements $i_k \in [n]$ are pairwise distinct. We call a $t$-design of index 1 a *sharply t-transitive* set.

We also call a $((1^{n-d+1}), d-1)$-clique $Y$ of $C_r \wr S_n$ a *d-code*. Hence, for any distinct $x, y \in Y$ we have that $x^{-1}y$ does not stabilise a $((1^{n-d+1}), d-1)$-tabloid. This implies that for any distinct $x, y \in Y$ there are at least $d$ elements $i \in [n]$ such that $x$ and $y$ map $(0, i)$ to different elements.

We can also give a geometric interpretation of $t$-designs.

**Example 5.6.9.** Let $G = C_2$ and $n = 3$. Then $W = C_2 \wr S_3$ is the symmetry group of a 3-dimensional cube. For a 1-design, we need to study the subgroup $H = C_2 \wr S_2$. Geometrically, $H$ is the stabiliser of a face of the cube. Thus, $Y \subseteq C_2 \wr S_3$ is a 1-design if and only if $Y$ is transitive on the six faces of a 3-dimensional cube.

We can extend this example to $t$-designs and arbitrary $n \in \mathbb{N}$. Then we consider the subgroup $H = (S_1)^t \times C_2 \wr S_{n-t} = C_2 \wr S_{n-t}$. Here, $H$ is the stabiliser of a chain of faces $F_0 \subseteq F_1 \subseteq \ldots \subseteq F_t$ of the cube where $F_i$ has dimension $n - t + i$. In other words, $H$ stabilises a $((1^t), n-t)$-flag (see Chapter 1). Thus, $Y \subseteq C_2 \wr S_n$ is a $t$-design if and only if $Y$ is transitive on the set of $((1^t), n-t)$-flags of the $n$-dimensional cube. Similar interpretations can be given for other regular polytopes.

Using the results from Section 5.3, we can characterise codes and designs in terms of the inner and dual distribution, respectively.

**Corollary 5.6.10.** *Let $Y$ be a subset of $C_r \wr S_n$ with inner distribution $(a_{\underline{\mu}})$ and dual distribution $(a'_{\underline{\lambda}})$. Then $Y$ is a $t$-design if and only if*

$$a'_{\underline{\lambda}} = 0 \quad \textit{for each } \underline{\lambda} \in \Lambda_n(C_r) \textit{ satisfying } n - t \leqslant \underline{\lambda}(0)_1 < n,$$

*and a $d$-code if and only if*

$$a_{\underline{\mu}} = 0 \quad \textit{for each } \underline{\mu} \in \Lambda_n(C_r) \textit{ satisfying } n - d + 1 \leqslant \underline{\mu}(0)'_1 - \underline{\mu}(0)'_2 < n.$$

*Proof.* The first part follows from Theorem 5.3.11. Note that for $\underline{\sigma} \in \Sigma_n(G)$ given by $\underline{\sigma}(\{0\}) = (1^t)$ and $\underline{\sigma}(G) = (n-t)$, we have $\underline{\sigma} \to \underline{\lambda}$ if and only if $\underline{\lambda}(0)_1 \geqslant n - t$, as was already observed before.

The second part follows from Theorem 5.4.4. We have to compute the set $C(\underline{\sigma})$ for $\underline{\sigma} = ((1^{n-d+1}), d-1)$. First, note that the conjugacy classes of $C_r \wr S_n$ appearing in the subgroup $C_r \wr S_{d-1}$ are exactly the classes indexed by the set $\Lambda_{d-1}(C_r)$. Moreover, the trivial subgroup $S_1$ only has the trivial conjugacy class $\underline{\mu} \in \Lambda_1(C_r)$ with $\underline{\mu}(0) = 1$ and $\underline{\mu}(g) = \varnothing$ otherwise. Hence, Theorem 5.4.3 gives that the elements of $C(\underline{\sigma})$ are precisely the elements $\underline{\mu} \in \Lambda_n(C_r)$ satisfying

$$\underline{\mu} = ((1^{n-d+1}), \varnothing, \ldots, \varnothing) \cup \underline{\lambda}$$

for some element $\underline{\lambda} \in \Lambda_{d-1}(C_r)$. Taking duals, we see that these are precisely the elements $\underline{\mu} \in \Lambda_n(G)$ with $\underline{\mu}(0)'_1 - \underline{\mu}(0)'_2 \geqslant n - d + 1$. □

Note that $d(x, y) = n - \theta(x^{-1}y)$ defines a metric on $C_r \wr S_n$. We can therefore define codes and a distance distribution in the usual way. For a non-empty subset $Y$ of $C_r \wr S_n$, we define the *distance distribution* to be the tuple $(A_i)_{0 \leqslant i \leqslant n}$, where

$$A_i = \frac{1}{|Y|} \left| \{(x, y) \in Y \times Y \mid d(x, y) = i\} \right|,$$

and the *dual distance distribution* of $Y$ to be the tuple $(A'_k)_{0 \leqslant k \leqslant n}$, where

$$A'_k = \sum_{i=0}^{n} C_k^{(1/r)}(n-i)A_i,$$

or equivalently

$$A'_k = \frac{1}{|Y|} \sum_{x,y \in Y} C_k^{(1/r)}(\theta(x^{-1}y)) = \frac{1}{|Y|} \sum_{x,y \in Y} C_k^{(1/r)}(n - d(x,y)). \tag{5.19}$$

We can characterise $t$-designs in terms of zeroes in the dual distance distribution.

**Proposition 5.6.11.** *Let $Y$ be a subset of $C_r \wr S_n$ with dual distance distribution $(A'_k)$ and let $t$ be an integer satisfying $1 \leqslant t \leqslant n$. If $Y$ is a $t$-design, then $A'_k = 0$ for all $k$ satisfying $1 \leqslant k \leqslant t$. Moreover, the converse also holds if $t \leqslant n/2$. That is, if $t \leqslant n/2$ and $A'_k = 0$ for all $k$ satisfying $1 \leqslant k \leqslant t$, then $Y$ is a $t$-design.*

*Proof.* First, assume that $Y \subseteq C_r \wr S_n$ is a $t$-design. From (5.19) and (5.16), we find that

$$A'_k = \frac{1}{|Y|} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \sum_{x,y \in Y} \xi_j(x^{-1}y). \tag{5.20}$$

By Theorem 5.3.10, the permutation character $\xi_j$ decomposes into those irreducible characters $\chi^{\underline{\lambda}}$ for which $\underline{\lambda}(0)_1 \geqslant n - j$. Moreover, since $\xi_j$ is a permutation character, it contains the trivial character $1_{C_r \wr S_n}$ as an irreducible constituent with multiplicity 1. From (3.15) and Corollary 5.6.10, it then follows that for all $j$ with $0 \leqslant j \leqslant t$, the inner sum in (5.20) is

$$\sum_{x,y \in Y} \xi_j(x^{-1}y) = \sum_{x,y \in Y} 1_{C_r \wr S_n}(x^{-1}y) = |Y|^2.$$

Hence, for all $k$ with $0 \leqslant k \leqslant t$, we have

$$A'_k = |Y| \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} = |Y|(1 + (-1))^k = |Y|\delta_{k,0}.$$

So we find that $A'_k = 0$ for all $k$ satisfying $1 \leqslant k \leqslant t$.

Now, for each $k$ with $0 \leqslant k \leqslant n/2$, we find from Theorem 5.6.6 that

$$C_k^{(1/r)}(\theta) = \sum_{\substack{\underline{\lambda} \in \Lambda_n(C_r) : \\ \underline{\lambda}(0)_1 = n-k}} \deg(\chi^{\underline{\lambda}})\, \chi^{\underline{\lambda}}.$$

Together with (5.19) and (3.15), we have that

$$A'_k = \frac{1}{|Y|} \sum_{\substack{\underline{\lambda} \in \Lambda_n(C_r) : \\ \underline{\lambda}(0)_1 = n-k}} \deg(\chi^{\underline{\lambda}}) \sum_{x,y \in Y} \chi^{\underline{\lambda}}(x^{-1}y) = \sum_{\substack{\underline{\lambda} \in \Lambda_n(C_r) : \\ \underline{\lambda}(0)_1 = n-k}} a'_{\underline{\lambda}}.$$

Suppose that $t \leqslant n/2$ and $A'_k = 0$ for all $k$ satisfying $1 \leqslant k \leqslant t$. Since the dual distribution $a'$ is non-negative, it follows that $a'_{\underline{\lambda}} = 0$ for all $\underline{\lambda} \in \Lambda_n(C_r)$ with $n - t \leqslant \underline{\lambda}(0)_1 < n$. Corollary 5.6.10 then implies that $Y$ is a $t$-design. □

We also have the following bounds.

**Corollary 5.6.12.** *Let $Y$ be a subset of $C_r \wr S_n$ and let $d$ and $t$ be the largest integers such that $Y$ is a $d$-code and a $t$-design. Then*

$$r^t\,(n)_t \leqslant |Y| \leqslant r^{n-d+1}\,(n)_{n-d+1}.$$

*Moreover, if equality holds in one of the bounds, then equality also holds in the other and this case happens if and only if $d = n - t + 1$.*

*Proof.* Use Theorem 5.4.5 for $\underline{\sigma} = ((1^t), n - t)$ and $\underline{\sigma} = ((1^{n-d+1}), d - 1)$. □

If we let $r = 1$, then the upper bound in Corollary 5.6.12 becomes a well-known bound for permutation codes (see Theorem 4.2.3). Moreover, the bounds in Corollary 5.6.12 can be achieved by sharply $t$-transitive sets. A construction for sharply 2-transitive sets is given in Section 5.7.

The last result is about distance distributions. It turns out that the distance distribution of a subset $Y$ of $C_r \wr S_n$ is uniquely determined, provided that $Y$ is a $t$-design and a $d$-code where $d \geqslant n - t$.

**Theorem 5.6.13.** *Suppose that $Y$ is a $t$-design and an $(n - t)$-code in $C_r \wr S_n$. Then the distance distribution $(A_i)$ of $Y$ satisfies*

$$A_{n-i} = \sum_{j=i}^{t} (-1)^{j-i} \binom{j}{i} \binom{n}{j} \left( \frac{|Y|}{r^j (n)_j} - 1 \right)$$

*for each $i \in \{0, 1, \dots, n - 1\}$.*

*Proof.* We have

$$A'_k = \sum_{i=0}^{n} C_k^{(1/r)}(i) A_{n-i}. \tag{5.21}$$

Using binomial inversion (5.10), we find from the definition of the Charlier polynomials that

$$r^j (x)_j = \sum_{k=0}^{j} \binom{j}{k} C_k^{(1/r)}(x). \tag{5.22}$$

Multiply both sides of (5.21) by $\binom{j}{k}$, sum over $k$, and use (5.22) to find that

$$\sum_{k=0}^{j} \binom{j}{k} A'_k = \sum_{i=0}^{n} A_{n-i}\, r^j\,(i)_j.$$

Since $Y$ is a $t$-design, we find from Proposition 5.6.11 that $A'_1 = \cdots = A'_t = 0$. So for every $j \in \{1, \ldots, t\}$, we have

$$A'_0 = \sum_{i=0}^{n} A_{n-i}\, r^j\, (i)_j.$$

Similarly, since $Y$ is an $(n-t)$-code, we have $A_1 = \ldots = A_{n-t-1} = 0$, and we find

$$A'_0 = A_0 r^j (n)_j + \sum_{i=0}^{t} A_{n-i}\, r^j\, (i)_j.$$

for every $j \in \{1, \ldots, t\}$. Moreover, we have $A_0 = 1$ and $A'_0 = |Y|$ and therefore

$$|Y| - r^j\, (n)_j = \sum_{i=0}^{t} A_{n-i}\, r^j\, (i)_j$$

for each $j \in \{1, 2, \ldots, t\}$. Divide by $j!\, r^j$ to obtain

$$\sum_{i=0}^{t} \binom{i}{j} A_{n-i} = \frac{|Y| - r^j\, (n)_j}{r^j\, j!} = \frac{(n)_j}{j!} \left( \frac{|Y| - r^j (n)_j}{r^j (n)_j} \right) = \binom{n}{j} \left( \frac{|Y|}{r^j\, (n)_j} - 1 \right)$$

for each $j \in \{1, 2, \ldots, t\}$. Now (5.11) gives the desired result. □

**Remark 5.6.14.** Note that for $Y = C_r \wr S_n$, we have that $Y$ is an $n$-design and a 1-code. Moreover, $A_{n-i}$ equals the number of elements in $C_r \wr S_n$ with precisely $i$ fixed points, hence $A_{n-i} = w_i$. In this case, Theorem 5.6.13 gives the well-known expression

$$w_i = \frac{n!\, r^{n-i}}{i!} \sum_{j=0}^{n-i} \frac{(-1)^j}{j!\, r^j}$$

for $i = 0, 1, \ldots, n-1$ after some elementary calculations. Additionally, we clearly have the value $w_n = 1$.

## 5.7 CONSTRUCTIONS AND APPLICATIONS

In this section, we give a simple construction of $t$-designs in $C_r \wr S_n$ using $t$-designs in $S_n$ and orthogonal arrays. This construction is reminiscent of constructions of designs in permutation groups using Gelfand pairs, see [6] and [33] for general groups, [46] for $S_n$, and [24] for $\mathrm{GL}(n, q)$. For a Gelfand pair $(G, H)$, such constructions use designs in $H$ and $G/H$ to construct designs in $G$ (more information on Gelfand pairs is given in Section 6.1). We also present an application of our results to the well-known prime power conjecture for finite projective planes.

It is well-known that $(C_r \wr S_n, S_n)$ is a Gelfand pair (since $C_r$ is abelian) that has applications to parking functions (see [1] for details). It also gives rise to an association scheme. The point set of this scheme is $(C_r \wr S_n)/S_n$, which we identify with $C_r^n$. We do not need to analyse this association scheme further but we remark that orthogonal arrays with $n$ columns and $r$ symbols can be interpreted as subsets of this scheme. We have already seen orthogonal arrays as designs in the Hamming scheme (see Theorem 3.2.13).

In what follows, we identify the elements of $C_r \wr S_n$ with pairs $(y, z)$ where $y \in C_r^n$ and $z \in S_n$. The following result is immediate.

**Proposition 5.7.1.** *Let $Y$ be a $t$-design in $S_n$ and let $D$ be an orthogonal array of strength $t$ with $n$ columns and $r$ symbols. Then*

$$\{(g, y) \mid g \in D, y \in Y\}$$

*is a $t$-design in $C_r \wr S_n$.*

*Proof.* Note that there are constantly many elements $(g, y)$ which map a $t$-tuple $((c_1, i_1), \ldots, (c_t, i_t))$ with all $i_k$ pairwise distinct to another $t$-tuple of this type. □

The special case that $Y = S_n$ in Proposition 5.7.1 was also stated by Ito [33, Section 8] in a more general context.

The construction of Proposition 5.7.1 is visualised in Figure 5.9 in the group $C_2 \wr S_3$. On the left, we have the permutation $\sigma = (123)$ written in one line notation as $\big(\sigma(1), \sigma(2), \sigma(3)\big)$ and in the middle we have an orthogonal array of strength 2 with 3 columns and 2 symbols.

For a permutation $\sigma \in S_n$, we create a generalised permutation for every row of the array. These are visualised on the right: every row is a generalised permutation, in our case a signed permutation. If we choose to let blue denote the element $+$ (that is, no sign change) and red the element $-$ (that is, a sign change), then the first row stands for the signed permutation with $1 \mapsto -2$, $2 \mapsto 3$ and $3 \mapsto 1$. The other rows can be interpreted similarly. Note that a row of coloured numbers uniquely determines a generalised permutation.

**Example 5.7.2.** If $r$ is a prime power and $r \geqslant n + 1$, then it is well-known that there are orthogonal arrays of strength $t$ with $n$ columns and $r$ symbols of minimum size $r^t$. These objects are equivalent to MDS codes in $H(n, r)$ of minimum distance $n - t + 1$. Hence, if $r \geqslant n + 1$ and there exists a $t$-design in $S_n$ of index 1 (namely a sharply $t$-transitive set in $S_n$), then by Proposition 5.7.1 there exists also a $t$-design of index 1 in $C_r \wr S_n$. For example, if $n$ is a prime

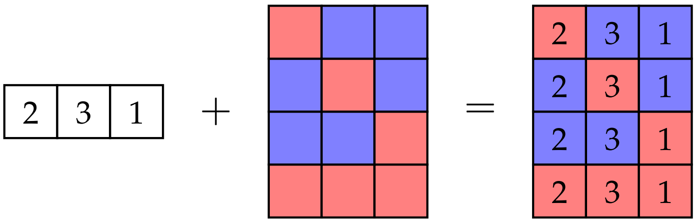



Figure 5.9: Construction of designs in $C_r \wr S_n$

power, then $\mathrm{AGL}(1, n)$ inside $S_n$ is a 2-design of index 1 and hence there exist 2-designs of index 1 in $C_r \wr S_n$ whenever $r$ and $n$ are prime powers with $r \geqslant n + 1$.

Not every $t$-design in $C_r \wr S_n$ comes from the construction of Proposition 5.7.1. For example, there is a 2-design of index 1 in $C_2 \wr S_5$ consisting of 80 elements (see Appendix a.1). If this example came from Proposition 5.7.1, then it would have to come from a sharply 2-transitive set in $S_5$ and an orthogonal array $A$ of strength 2 and index 1. Then $A$ would have to have precisely 4 rows and 5 columns over the alphabet $\{0, 1\}$. However, it is easy to see that such an orthogonal array does not exist (see a.1).

Now note that by Corollary 5.6.12, each $t$-design $Y$ in $C_r \wr S_n$ must satisfy $|Y| \geqslant r^t (n)_t$. Using the results of [37], we now show that there are always $t$-designs in $C_r \wr S_n$ almost as small as this lower bound.

**Corollary 5.7.3.** *Let $r, n, t$ be integers satisfying $r \geqslant 2$, $n \geqslant 1$, and $1 \leqslant t \leqslant n$. Then there exists a $t$-design $Y$ in $C_r \wr S_n$ satisfying $|Y| \leqslant (crn)^{ct}$ for some universal constant $c > 0$.*

*Proof.* From [37, Theorems 1.1 and 1.6], we find that there exists an orthogonal array $D$ of strength $t$ with $n$ columns and $r$ symbols satisfying $|D| \leqslant \left(\frac{c_1 rn}{t}\right)^{c_1 t}$ and a $t$-design $Y$ in $S_n$ satisfying $|Y| \leqslant (c_2 n)^{c_2 t}$ for some universal constants $c_1, c_2 > 0$. The statement of the corollary then follows from Proposition 5.7.1. □

Note that Corollary 5.7.3 also gives the existence of more general designs in $C_r \wr S_n$, since in view of Theorem 5.5.11, every $t$-design is also a $(\sigma, \rho)$-design, where $(\sigma, \rho)$ is a double partition of $n$ such that $\rho_1 \geqslant n - t$.

Now, we give an application of our results to finite projective planes. For an introduction to projective planes, we refer the reader to [31]. We make use of two well-known results on the existence of projective planes.

**Theorem 5.7.4** (Bose [9])**.** *There exists a projective plane of order $n$ if and only if there exists a complete set of $n - 1$ mutually orthogonal Latin squares of order $n$.*

It is well-known that there exists a complete set of mutually orthogonal Latin squares of order $n$ if and only if there exists an orthogonal array of strength 2 with $n+1$ columns, $n$ symbols and index 1 (see for example [61, Theorem 6.38]). Thus, the existence of a projective plane of order $n$ is equivalent to the existence of a 2-design in the Hamming scheme $H(n+1, n)$ of index 1.

It is also well-known that the existence of a finite projective plane of order $n$ is connected to the existence of a sharply 2-transitive set in $S_n$, so a 2-design in $S_n$ of index 1.

**Theorem 5.7.5** (Hall [29])**.** *There exists a projective plane of order $n$ if and only if there exists a sharply 2-transitive subset of $S_n$.*

Using Proposition 5.7.1, we can construct a 2-design in $C_r \wr S_n$ from a 2-design in $S_n$ and an orthogonal array of strength 2. This gives the following theorem.

**Theorem 5.7.6.** *Let $n$ be a positive integer. If there is no 2-design of index 1 in $C_{n-1} \wr S_n$, then there is no projective plane of order $n-1$ or there is no projective plane of order $n$.*

*Proof.* We show that the existence of a projective plane of order $n-1$ and order $n$ implies the existence of a 2-design of index 1 in $C_{n-1} \wr S_n$.

A projective plane of order $n$ gives rise to a 2-design in $S_n$ of index 1. A projective plane of order $n-1$ gives rise to an orthogonal array of strength 2 with $n$ columns and $n-1$ symbols of index 1. Now Proposition 5.7.1 gives a 2-design in $C_{n-1} \wr S_n$ of index 1. □

If $n$ is a prime power, then there exists a projective plane of order or $n$. In this case, the non-existence of a 2-design in $C_{n-1} \wr S_n$ implies the non-existence of a projective plane of order $n-1$. Similarly, if $n-1$ is a prime power, then the non-existence of a 2-design in $C_{n-1} \wr S_n$ implies the non-existence of a projective plane of order $n$. We summarise our findings in the following corollary.

**Corollary 5.7.7.** *Let $q$ be a prime power. If there is no 2-design of index 1 in $C_{q-1} \wr S_q$, then there is no finite projective plane of order $q-1$. If there is no 2-design of index 1 in $C_q \wr S_{q+1}$, then there is no finite projective plane of order $q+1$.*

Note that it is possible that there exist 2-designs of index 1 in $C_{n-1} \wr S_n$ that do not come from the construction of Proposition 5.7.1 like the example in a.1 in $C_2 \wr S_5$. In this case, Theorem 5.7.6 cannot rule out the existence of a finite projective plane. More detailed analysis of the structure of transitive sets in $C_{n-1} \wr S_n$ is needed to explore this connection further.

## 5.8 OPEN PROBLEMS

We list some open problems for further research that the reader might be interested to work on.

(1) Prove Conjecture 5.5.14 or Conjecture 5.5.15.

(2) Find geometric interpretations for $\underline{\sigma}$-transitive sets in the groups $G \wr S_n$ where $G$ is not cyclic.

(3) Can Corollary 5.7.7 be used to show the non-existence of a finite projective plane? In order to rule out the case $n = 12$, one would have to work in the group $C_{11} \wr S_{12}$ or $C_{12} \wr S_{13}$.

(4) Investigate $T$-designs for other sets $T$ that were not dealt with in this thesis.

# 6 DESIGNS OF PERFECT MATCHINGS

We now turn to a different association scheme – the *perfect matching association scheme*. Although this association scheme is not a conjugacy class association scheme like the ones studied in Chapter 5, it has many striking similarities. This is because the perfect matching association scheme comes from a so-called *Gelfand pair*. It provides us with *zonal spherical functions* that have properties similar to irreducible characters.

Recall that 1-factorisations and hyperfactorisations are examples of partition systems. We generalise the notion of $s$-$(2, n)$ partition systems to *$\lambda$-factorisations* by replacing the parameter $s$ with a partition $\lambda$ and use the theory of association schemes to investigate them. In an $s$-$(2, n)$ partition system, any set of $s - 1$ subsets of size 2 appears a constant number of times. The counting involved does not give rise to restrictions on the parameters $s$ and $n$ (see [12, Theorem 7.2]). We show that similar counting arguments work for $\lambda$-factorisations and give rise to non-trivial divisibility conditions by comparing different partitions $\lambda$.

In this chapter, we first give a brief introduction to Gelfand pairs. Then, we introduce the association scheme of perfect matchings and give formulas for the $Q$-numbers. We define $\lambda$-factorisations and show that they are $T$-designs in the sense of Delsarte. Using our characterisation, we compare $\lambda$-factorisations for different partitions $\lambda$, obtaining a result that can be viewed as an analogue of the Livingstone–Wagner theorem. Furthermore, we derive divisibility conditions that give rise to restrictions on the parameter $n$ and various non-existence results. We end with a survey of $\lambda$-factorisations for small $n$ and some open problems.

## 6.1 GELFAND PAIRS AND ASSOCIATION SCHEMES

We start by presenting some well-known theory of Gelfand pairs. An excellent treatment of Gelfand pairs is given in [45] from which the statements about Gelfand pairs in this chapter are taken. In fact, the conjugacy class scheme of the generalised symmetric group that we studied in Chapter 5 is also a Gelfand pair. Hence, it is not surprising that these association schemes behave similarly in many ways.

### 6.1.1 *Group Algebras*

The theory of complex group algebras will prove useful to derive results about association schemes. Denote by $\mathbb{C}[G]$ the complex group algebra of $G$. It is the vector space $\mathbb{C}G$ of complex functions on $G$ together with convolution as multiplication. The group algebra $\mathbb{C}[G]$ is generated by the basis $(e_\sigma)_{\sigma\in G}$. We can think of a function $f : G \to \mathbb{C}$ as the formal sum

$$f = \sum_{\sigma\in G} f(\sigma)e_\sigma.$$

Going the other way, we can think of an element $f \in \mathbb{C}[G]$ as a function from $G$ to $\mathbb{C}$ where $f(\sigma)$ is the coefficient of $e_\sigma$ when $f$ is written in terms of the basis $(e_\sigma)_{\sigma\in G}$. Since every character $\chi$ of $G$ is a function from $G$ to $\mathbb{C}$, every character is an element of the group algebra $\mathbb{C}[G]$. The multiplication of two elements $f, g \in \mathbb{C}[G]$ is defined as

$$(f * g)(\sigma) = \sum_{\tau\in G} f(\sigma\tau^{-1})g(\tau)$$

for every $\sigma \in G$. Let $H$ be a subgroup of $G$. Denote by $e_H$ the element

$$e_H = \frac{1}{|H|}\sum_{h\in H} e_h.$$

Then $e_H * e_H = e_H$ and $e_H$ is an idempotent of the group algebra. As a function from $G$ to $\mathbb{C}$, we have $e_H(\sigma) = |H|^{-1}$ if $\sigma \in H$ and $e(\sigma) = 0$ otherwise. Inside $\mathbb{C}[G]$, we can investigate the subalgebra $e_H * \mathbb{C}[G] * e_H$. As convolution of a function $f : G \to \mathbb{C}$ with $e_H$ roughly corresponds to averaging $f$ over $H$ (from the left or from the right), we find that $e_H * \mathbb{C}[G] * e_H$ is the algebra of functions $f : G \to \mathbb{C}$ such that

$$f(h\sigma h') = f(\sigma) \quad \text{for all } h, h' \in H \text{ and } \sigma \in G,$$

that is, the functions that are both left- and right-invariant under multiplication with elements of $H$. Thus, $e_H * \mathbb{C}[G] * e_H$ is the algebra of functions from $G$ to $\mathbb{C}$ that are constant on the double cosets $H\sigma H$ of $G$. This algebra will be denoted by $\mathbb{C}[H\backslash G/H]$. It inherits the inner product $\langle\cdot,\cdot\rangle$ from $\mathbb{C}G$.

### 6.1.2 *Gelfand Pairs*

We refer the reader to [45, Ch. VII.1] for an introduction to Gelfand pairs. A (finite) *Gelfand pair* is a pair of a group $G$ and a subgroup $H \leqslant G$ such that the group algebra $\mathbb{C}[H\backslash G/H]$ is commutative. Equivalently, $(G, H)$ is a

Gelfand pair if and only if the induced character $1\uparrow_H^G$ of $H$ is multiplicity-free (by Schur's Lemma). We will only work with finite Gelfand pairs, hence the following theory assumes that $G$ is finite.

Every Gelfand pair $(G, H)$ gives rise to an association scheme by considering the transitive action of $G$ on the (left) cosets of $H$, and taking the orbits of $G$ on pairs for the relations. Letting $G = S_{2n}$ and $H = S_2 \wr S_n$, we have that

$$1\uparrow_{S_2 \wr S_n}^{S_{2n}} = \sum_{\lambda \vdash n} \chi^{2\lambda},$$

hence $(S_{2n}, S_2 \wr S_n)$ is a Gelfand pair. We refer the reader to [45, Ch. VII.2] for an excellent treatment of this Gelfand pair. Following [45], we drop the factor of $\frac{1}{|G|}$ in the definition of the inner product of $\mathbb{C}[G]$ when working in the Gelfand pair $(S_{2n}, S_2 \wr S_n)$ and use the inner product given by

$$\langle f, g \rangle = \sum_{\sigma \in S_{2n}} f(\sigma)\overline{g(\sigma)}$$

for every $f, g \in \mathbb{C}[S_{2n}]$.

In spirit with the theory of Coxeter groups, we denote the subgroup $S_2 \wr S_n$ by $B_n$. We have that $|B_n| = 2^n n!$. The subgroup $B_n$ is not a normal subgroup of $S_{2n}$, so $S_{2n}/B_n$ is not a group and hence we cannot work directly with characters and permutation representations. However, every Gelfand pair has so-called *zonal spherical functions* that behave very similarly to irreducible characters, and many calculations with irreducible characters in a group can be mimicked with zonal spherical functions in a Gelfand pair.

The zonal spherical functions of the Gelfand pair $(S_{2n}, B_n)$ are functions from $S_{2n}$ to the complex numbers. They are indexed by the partitions of $n$. In the group algebra $\mathbb{C}[S_{2n}]$, the zonal spherical functions $\omega^\lambda$ are given by

$$\omega^\lambda = \chi^{2\lambda} * e_{B_n} = e_{B_n} * \chi^{2\lambda}$$

where $\chi^{2\lambda}$ is the irreducible character of $S_{2n}$ indexed by the partition $2\lambda$. From the definition, it is immediate that the zonal spherical functions are constant on the double cosets $B_n \sigma B_n$ so we find that $\omega^\lambda \in \mathbb{C}[B_n \backslash S_{2n} / B_n]$. The double cosets are indexed by partitions of $n$ and we denote the value of the zonal spherical function $\omega^\lambda$ on the double coset of type $\rho$ by $\omega^\lambda_\rho$. Explicitly, the value of $\omega^\lambda$ on an element $\sigma \in S_{2n}$ is given by

$$\omega^\lambda(\sigma) = \frac{1}{|B_n|} \sum_{b \in B_n} \chi^{2\lambda}(\sigma b).$$

The zonal spherical functions of the Gelfand pair $(S_{2n}, S_2 \wr S_n)$ have the following properties.

**Theorem 6.1.1** ([45, Ch. VII (1.4)]). *Let $n \in \mathbb{N}$ and $\lambda, \mu \vdash n$. Then,*

(i) $\omega^{(n)} = 1$,

(ii) $\omega^\lambda(\sigma^{-1}) = \omega^\lambda(\sigma)$ *for all* $\sigma \in S_{2n}$,

(iii) $\langle \omega^\lambda, \omega^\mu \rangle = \delta_{\lambda\mu}(2n)!\chi^{2\lambda}(1)^{-1}$,

(iv) $\omega^\lambda * \omega^\mu = \delta_{\lambda\mu} c_\lambda \omega^\lambda$ *for some* $c_\lambda \neq 0$,

(v) *The zonal spherical functions* $(\omega^\lambda)_{\lambda \vdash n}$ *form an orthogonal basis of* $\mathbb{C}[B_n \backslash S_{2n} / B_n]$.

So the zonal spherical functions can be thought of as analogues of the irreducible characters of a group.

**Remark 6.1.2.** The conjugacy class scheme investigated in Chapter 5 is a Gelfand pair. To see this, let $G$ be a finite group and consider the direct product $G \times G$. The *diagonal group*

$$\operatorname{diag}(G) = \{(g, g) \in G \times G \mid g \in G\}$$

is a subgroup of $G$. It is easy to see that $(G \times G, \operatorname{diag}(G))$ is a Gelfand pair. First, notice that the irreducible characters of $G \times G$ are precisely the products $\chi_i \cdot \chi_j$ where $\chi_i$ and $\chi_j$ are irreducible characters of $G$. Now we calculate using Frobenius reciprocity

$$\begin{aligned}\langle 1 \uparrow_{\operatorname{diag}(G)}^{G \times G}, \chi_i \cdot \chi_j \rangle = \langle 1_{\operatorname{diag}(G)}, (\chi_i \cdot \chi_j) \downarrow_{\operatorname{diag}(G)}^{G \times G} \rangle &= \frac{1}{|G|} \sum_{g \in G} \overline{\chi_i(g) \cdot \chi_j(g)} \\ &= \frac{1}{|G|} \sum_{g \in G} \overline{\chi_i(g)} \cdot \overline{\chi_j(g)} = \langle \overline{\chi_i}, \chi_j \rangle.\end{aligned}$$

If $\chi_i$ is an irreducible character of $G$, so is $\overline{\chi_i}$, hence we find from Theorem 2.1.23 that

$$\langle \overline{\chi_i}, \chi_j \rangle = \begin{cases} 1, & \chi_j = \overline{\chi_i}, \\ 0, & \text{otherwise.} \end{cases}$$

It follows that the induced character $1 \uparrow_{\operatorname{diag}(G)}^{G \times G}$ is multiplicity-free and hence $(G \times G, \operatorname{diag}(G))$ is a Gelfand pair.

The base set of the association scheme that arises from the Gelfand pair $(G \times G, \operatorname{diag}(G))$ is the set of left cosets of $\operatorname{diag}(G)$. It is clear that each coset contains a unique element of the form $(x, e)$ for some $x \in G$. Hence, we can identify the base set with the group $G$ this way.

The relations of the association scheme are the orbits on pairs. Let $(x, e)$ and $(y, e)$ be representatives of two cosets of $\mathrm{diag}(G)$ where $x, y \in G$. Then the orbit of the pair $\big((x, e), (y, e)\big)$ consists of all pairs of the form

$$(g, h) \cdot \big((x, e), (y, e)\big) = \big((gx, h), (gy, h)\big)$$

for $(g, h) \in G \times G$. Now $(gx, h)$ represents the same coset as $(gxh^{-1}, e)$ and $(gy, h)$ represents the same coset as $(gyh^{-1}, e)$. So in terms of our identification of cosets with elements of $G$, the orbit of the pair $(x, y)$ is given by

$$R_{(x,y)} = \{(gxh^{-1}, gyh^{-1}) \in G \times G \mid g, h \in G\}. \tag{6.1}$$

We show that this is a relation of the conjugacy class scheme.

First, notice that every element $(gxh^{-1}, gyh^{-1}) \in R_{(x,y)}$ satisfies

$$(gxh^{-1})^{-1}gyh^{-1} = hx^{-1}g^{-1}gyh^{-1} = hx^{-1}yh^{-1}.$$

That is, for every element $(\tilde{x}, \tilde{y}) \in R_{(x,y)}$, it holds that $\tilde{x}^{-1}\tilde{y}$ is conjugate to $x^{-1}y$. It follows that $R_{(x,y)} \subseteq R_C$ where $R_C$ is the relation of the conjugacy class scheme corresponding to the conjugacy class $C$ of $x^{-1}y$.

Conversely, let $C$ be a conjugacy class of $G$, let $c \in C$ and consider the relation $R_C$ of the conjugacy class scheme. We show that $R_C \subseteq R_{(e,c)}$.

First, observe that if $\tilde{c} \in G$ is conjugate to $c$, that is $\tilde{c} = gcg^{-1}$ for some $g \in G$, then $(e, \tilde{c}) \in R_{(e,c)}$. This follows from (6.1) by setting $x = e$, $y = c$ and $g = h$. Hence, $R_{(e,c)} = R_{(e,\tilde{c})}$ for every $\tilde{c} \in C$.

Now, notice that if $(x, y) \in G \times G$ is a pair with $x^{-1}y = \tilde{c}$ for some $\tilde{c} \in C$, then $(x, y) \in R_{(e,\tilde{c})} = R_{(e,c)}$. This follows from (6.1) by setting $g = x^{-1}$ and $h = e$. Hence, we find that the relation $R_{(e,c)}$ contains every pair $(x, y) \in G \times G$ with $x^{-1}y \in C$, so $R_C \subseteq R_{(e,c)}$.

Thus, the association scheme from the Gelfand pair $(G \times G, \mathrm{diag}(G))$ turns out to be the conjugacy class scheme of $G$ and the zonal spherical functions turn out to be the irreducible characters of $G$ (up to a scalar). This explains the similarity of the properties in Theorem 6.1.1 to the properties of irreducible characters.

### 6.1.3 *The Perfect Matching Association Scheme*

We now describe the association scheme obtained from the Gelfand pair $(S_{2n}, B_n)$ in more detail. (Some of what we cover below can be found in [41, Section 4].) The base set of the association scheme is the coset space $S_{2n}/B_n$. As $B_n$ is the stabiliser of a perfect matching of $2n$ points, we find that the cosets of $B_n$ are in bijection with the perfect matchings of the complete graph on the vertices $1, \dots, 2n$. We can thus think of cosets of $B_n$ as perfect matchings and vice versa.

Denote by $m^*$ the perfect matching of the vertices $2i-1$ and $2i$ for $i = 1, \dots, n$. This can be thought of as a *base matching* or *identity matching*. It corresponds to the coset of the identity, so $m^* = B_n$. The relations of the scheme are indexed by partitions of $n$. Two perfect matchings $m, \widetilde{m}$ are in relation $R_\lambda$ (denoted by $r(m, \widetilde{m}) = \lambda$) if the (disjoint) union of $m$ and $\widetilde{m}$ consists of even cycles of lengths $2\lambda_1, 2\lambda_2, \dots$ . Figure 6.1 gives an example.

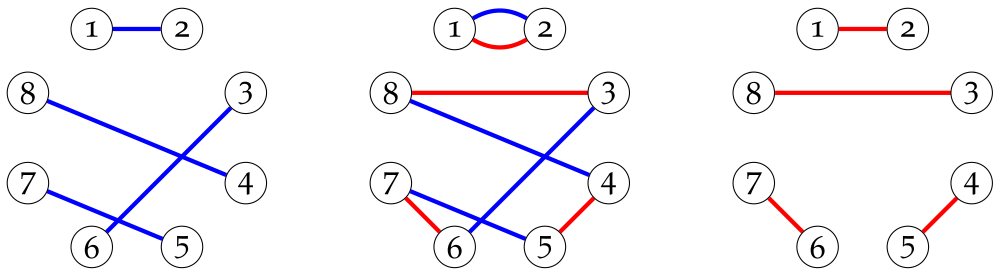
The set

$$\Omega_\lambda = \{\sigma B_n \mid r(m^*, \sigma B_n) = \lambda\}$$

is called the $\lambda$-*sphere*. We have that

$$r(\sigma B_n, \tau B_n) = \lambda \iff r(B_n, \sigma^{-1}\tau B_n) = \lambda \iff \sigma^{-1}\tau B_n \in \Omega_\lambda.$$

If $\sigma \in S_{2n}$ such that $\sigma B_n \in \Omega_\lambda$, we say that $\sigma$ has *coset type* $\lambda$. This is justified by the fact that $\sigma$ and $\tau$ have the same coset type if and only if $\tau \in B_n \sigma B_n$ (see [45, Ch. VII (2.1)]). The double cosets are indexed by partitions of $n$. The reader should note that the zonal spherical functions are constant on the $\lambda$-spheres just like the irreducible characters of a group are constant on the conjugacy classes.




$C_6 \cup C_2 \longrightarrow$ type (31)

Figure 6.1: Two perfect matchings and their union

### 6.1.4 *Eigenvalues of the Perfect Matching Association Scheme*

Knowledge of the (dual) eigenvalues of an association scheme is of crucial importance when investigating the dual distribution and designs. Godsil and Meagher [27] have computed the eigenvalues of an association scheme obtained from a finite Gelfand pair. We repeat the result here.

**Theorem 6.1.3** ([27, Cor. 13.8.2])**.** *Let $(G, H)$ be a finite Gelfand pair and $\mathbb{A}$ be the association scheme obtained from the action of $G$ on the left cosets of $H$. The matrix idempotents of $\mathbb{A}$ are $E_\phi$ where the entries are given by*

$$E_\phi(x_iH, x_jH) = \frac{\phi(1)}{|G|} \sum_{\sigma \in H} \phi(x_i \sigma x_j^{-1})$$

*and $\phi$ is an irreducible constituent of $1 \uparrow_H^G$.*

**Corollary 6.1.4.** *The idempotents of the perfect matching association scheme are given by*

$$E_\mu(x_iB_n, x_jB_n) = \frac{\chi^{2\mu}(1)}{(2n)!} \sum_{\sigma \in B_n} \chi^{2\mu}(x_i \sigma x_j^{-1}).$$

*The $(x_iB_n, x_jB_n)$-entry of $E_\mu$ is equal to $Q_\mu(\rho)/(2n-1)!!$ where $r(x_iB_n, x_jB_n) = \rho$, equivalently $x_i^{-1}x_j$ has coset type $\rho$. We obtain*

$$Q_\mu(\rho) = \chi^{2\mu}(1)\omega_\rho^\mu.$$

*Proof.* Notice that

$$\begin{aligned}
\frac{\chi^{2\mu}(1)}{(2n)!} \sum_{\sigma \in B_n} \chi^{2\mu}(x_i \sigma x_j^{-1}) &= \frac{\chi^{2\mu}(1)}{(2n)!} \sum_{\sigma \in B_n} \chi^{2\mu}(x_j^{-1} x_i \sigma) \\
&= \frac{\chi^{2\mu}(1)|B_n|}{(2n)!} \omega^\mu(x_j^{-1}x_i) \\
&= \frac{\chi^{2\mu}(1)}{(2n-1)!!} \omega^\mu(x_j^{-1}x_i).
\end{aligned}$$

The value $\omega^\mu$ only depends on the coset type of $x_j^{-1}x_i$ which has the same coset type as $(x_j^{-1}x_i)^{-1} = x_i^{-1}x_j$. □

Using this corollary and Equation (3.9), we obtain a formula for the dual distribution. The dual distribution of a non-empty set $Z \subseteq S_{2n}/B_n$ is given by

$$a'_\mu(Z) = \sum_{\rho \vdash n} Q_\mu(\rho) a_\rho(Z) = \chi^{2\mu}(1) \sum_{\rho \vdash n} \omega_\rho^\mu a_\rho(Z).$$

Hence,

$$a'_\mu(Z) = 0 \iff \sum_{\rho \vdash n} \omega_\rho^\mu a_\rho(Z) = 0.$$

## 6.2 DESIGNS

In this section, we introduce a generalisation of hyperfactorisations of $K_{2n}$.

**Definition 6.2.1.** Let $\lambda \vdash n$ and consider all set partitions of shape $2\lambda$. A set $D \neq \varnothing$ of perfect matchings is called a *$\lambda$-factorisation* if there exists a constant $c > 0$ such that for every set partition $P$ of shape $2\lambda$ there are exactly $c$ matchings $m \in D$ such that $m$ refines $P$. The constant $c$ is called the *index* of $D$.

**Example 6.2.2.** For every $n$, an $(n-1,1)$-factorisation with index 1 is just a 1-factorisation, and an $(n-2,1,1)$-factorisation is a hyperfactorisation. Cameron's examples [12] arising from hyperovals of projective planes are $(n-2,1,1)$-factorisations of index 1 where $2n = 2^a + 2$ for some integer $a \geqslant 3$. Indeed, Cameron's $s$-$(2,2n)$ partition systems are precisely the $(n-s,1,1,\dots,1)$-factorisations with index 1.
For every $n$ and $t \leqslant n/2$, an $(n-t,t)$-factorisation is a set $D \neq \varnothing$ of perfect matchings such that for every subset $S \subseteq \{1,\dots,2n\}$ of size $2t$, there are a constant number of perfect matchings in $D$ that have $t$ edges that all lie inside of $S$. Hence, every subset of size $2t$ is covered constantly often by the perfect matchings of $D$.

A *hyperoval* $\mathcal{O}$ of a projective plane $\Pi$ of order $q$ is a set of $q+2$ points such that every line of $\Pi$ intersects $\mathcal{O}$ in 0 or 2 points. Each point $P$ not in $\mathcal{O}$ defines a fixed-point-free permutation of $\mathcal{O}$, taking an element $X$ to the unique element $X^P$ of $\mathcal{O}\backslash\{X\}$ lying on the line spanned by $P$ and $X$. The set of $q^2-1$ permutations of $\mathcal{O}$ arising this way gives rise to an *abstract hyperoval*. Indeed, each permutation is a complete product of 2-cycles, and so the set of perfect matchings that arise is identical to Cameron's construction. However, not all abstract hyperovals arise from a hyperoval.

**Example 6.2.3.** Abstract hyperovals were introduced by J. G. Thompson in [63]. An *abstract hyperoval* $A(X)$ on a finite set $X$ is a set of fixed-point-free involutions on $X$ with the following property: for any four elements $a,b,c,d \in X$, there is a unique $u \in A(X)$ such that $a^u = b$ and $c^u = d$. If $|X| = n+2$, then we say that $A(X)$ has *order $n$*, and we see from the definition that $n$ is even and $|A(X)| = n^2-1$. De Clerck [16] showed that an abstract hyperoval of order $2s$ is equivalent to a partial geometry with parameters $\mathrm{pg}(s, 2s-2, s-1)$. We refer to [17] and [50] for more on this connection.

Now, a perfect matching defines a fixed-point-free involution, and vice versa, because a fixed-point-free involution is a complete product of disjoint 2-cycles. So if $n$ is even, a 2-$(2,n+2)$ partition system is a set of $n^2-1$ perfect matchings of $K_{n+2}$ such that every pair of disjoint edges $\{a, b\}$ and $\{c, d\}$ lies in exactly one perfect matching. Thus, we see that abstract hyperovals of order $n$ and 2-$(2, n+2)$ partition systems are equivalent objects. This observation generalises [12, Theorem 7.3], and can also be deduced from [17, Chapter 7] and [25, Lemma 1].

For $n = 2, 4$, the only abstract hyperovals are obtained by taking the set of all fixed-point-free involutions. There is no example for $n = 6$ (see [48]) and there are precisely two examples for $n = 8$ (see also [48]). So there are only two 2-$(2, 10)$ partition systems up to equivalence. In our language, there are two $(311)$-factorisations of index 1. We will look at related examples in Section 6.4. We remark that Thompson [63] explored the use of the representation theory of $S_{2n}$ to derive restrictions on the existence of abstract hyperovals, and we take this viewpoint further and apply this theory to $\lambda$-factorisations.

Our goal is to show the following theorem that characterises $\lambda$-factorisations as $T$-designs in an association scheme.

**Theorem 6.2.4.** *Let $D \subseteq S_{2n}/B_n$ be a non-empty set of perfect matchings and $(a'_\mu)$ be its dual distribution. Then*

$$D \text{ is a } \lambda\text{-factorisation} \iff a'_\mu = 0 \text{ for all } \mu \vdash n \text{ with } \lambda \trianglelefteq \mu \neq (n).$$

### 6.2.1 *Antidesigns*

We will prove Theorem 6.2.4 using the notion of *antidesigns*.

**Definition 6.2.5.** Let $(X, R)$ be an association scheme and $A, D \subseteq X$ be non-empty subsets of $X$ with dual distributions $(a_k'^A)_{k=0,\dots,d}$ and $(a_k'^D)_{k=0,\dots,d}$, respectively.

(a) The pair $(A, D)$ is called *design-orthogonal*, if $a_k'^A \cdot a_k'^D = 0$ for every $k > 0$.

(b) Let $T = \{k \in \{1, \dots, d\} \mid a_k'^A \neq 0\}$. Then we call $A$ a *$T$-antidesign*.

It is clear from the definition that if the pair $(A, D)$ is design-orthogonal and $A$ is a $T$-antidesign, then $D$ must be a $T$-design. Hence, we can characterise $T$-designs using $T$-antidesigns.

While a $T$-design can be thought of as a subset that 'evenly covers' all possibilities, a $T$-antidesign can quite literally be thought of as the opposite: A subset that only covers one single possibility. Let us give an example in the symmetric group.

**Example 6.2.6.** Consider the symmetric group $S_4$ acting on the set $\{1, 2, 3, 4\}$. The cyclic group $Y = \langle(1, 2, 3, 4)\rangle$ is a transitive subgroup of $S_4$. In other words, it is a (31)-design in the symmetric group. The dual distribution $a'$ of $Y$ is

$$a' = (4, 0, 8, 12, 0)$$

where the dual distribution is indexed by the partitions (4), (31), (22), (211), (1111) in that order.

A transitive group moves every element $x$ to every element $y$ constantly often. The opposite of that is moving an element $x$ to a fixed element $y$ and not to any others. The group elements that do this form a coset of the stabiliser of $x$.

The stabiliser of 1 under the action of $S_4$ on $\{1, 2, 3, 4\}$ is $A = S_{\{2,3,4\}} \cong S_3$. It is a Young subgroup of shape (31). The cosets

$$(1, 2) \circ A, \quad (1, 3) \circ A, \quad (1, 4) \circ A$$

are the sets of permutations that move the element 1 to 2, 3, 4, respectively.

The dual distribution $a'$ of $A$ (and in fact the dual distribution of all of its cosets) is

$$a' = (6, 18, 0, 0, 0).$$

Hence, $A$ is a $\{(31)\}$-antidesign as $a'_{(31)} \neq 0$.

Notice that the transitive subset $C_4$ (which is a $T$-design for $T = \{(31)\}$) has a 0 in the entry (31) where the dual distribution of the $\{(31)\}$-antidesign is non-zero.

This is true in general. It turns out (see [21]) that if an association scheme admits an automorphism group that acts transitively on each relation $R_i$, then a pair $(A, D)$ is design-orthogonal if and only if the intersection $|D \cap \varphi(A)|$ is constant when $\varphi$ varies over all automorphisms of $(X, R)$. This is the reason why considering antidesigns and their dual distribution is a useful technique to study designs. The approach of using antidesigns is also studied in detail in [53].

Applied to the association scheme of perfect matchings, we find that $S_{2n}$ acts transitively on each of the relations $R_\mu$. Let $\lambda \vdash n$ and let $P_\lambda$ be a set partition of shape $2\lambda$. Consider the set $A$ of all perfect matchings that refine $P_\lambda$. A $\lambda$-factorisation $D$ of index $c$ intersects $A$ in precisely $c$ elements. Note that $S_{2n}$ acts transitively on the set partitions of shape $2\lambda$.

From the definition of a $\lambda$-factorisation, we find that $|D \cap \varphi(A)|$ is constant for all $\varphi \in S_{2n}$ as $\varphi(A)$ is the set of all perfect matchings that refine the set partition $\varphi(P_\lambda)$. Thus, the pair $(A, D)$ is design-orthogonal. If $A$ is a $T$-antidesign, then $D$ is a $T$-design. Hence, it suffices to compute the set $T$ such that $A$ is a $T$-antidesign.

We find that $A$ is the set of left cosets of $B_n$ with representatives in $S_{2\lambda}B_n$, where $S_{2\lambda}$ is a Young subgroup of shape $2\lambda$. The size of $A$ in the association scheme $S_{2n}/B_n$ is

$$|A| = \frac{|S_{2\lambda}B_n|}{|B_n|} = \frac{|S_{2\lambda}|}{|S_{2\lambda} \cap B_n|} = \frac{|S_{2\lambda}|}{|B_\lambda|} = \frac{(2\lambda)!}{2^n \lambda!}$$

where $B_\lambda = B_{\lambda_1} \times B_{\lambda_2} \times \cdots \times B_{\lambda_{l(\lambda)}}$. We have that $|B_\lambda| = 2^n\lambda!$.

First, we compute the inner distribution of $A$.

**Lemma 6.2.7.** *The inner distribution $a_\rho(A)$ is given by*

$$a_\rho(A) = \frac{1}{2^n\lambda!}|S_{2\lambda} \cap B_n x_\rho B_n|$$

*where $x_\rho \in S_{2n}$ is any element of coset type $\rho$.*

*Proof.* First, recall that $r(\sigma B_n, \tau B_n) = \rho$ if and only if the coset type of $\sigma^{-1}\tau$ is $\rho$. Since every matching in $A$ corresponds to a left coset $\sigma B_n$ and all elements of $\sigma B_n$ have the same coset type, we get the number of pairs $(\sigma B_n, \tau B_n)$ with $r(\sigma B_n, \tau B_n) = \rho$ by counting the number of pairs $(\sigma, \tau) \in (S_{2\lambda}B_n)^2$ such that $\sigma^{-1}\tau$ is of coset type $\rho$ and dividing the result by $|B_n|^2$. Write

$$S_{2\lambda}B_n = \bigcup_{r \in R} S_{2\lambda} r$$

where $R \subseteq B_n$ is a complete set of representatives of the decomposition of $S_{2\lambda}B_n$ into right cosets of $S_{2\lambda}$. Then every element of $z \in A$ can be uniquely written as $z = \sigma r$ with $\sigma \in S_{2\lambda}$ and $r \in R$. Note that $|R| = |B_n : S_{2\lambda} \cap B_n| = |B_n : B_\lambda| = |A||B_n|/|S_{2\lambda}|$. Counting the pairs in the symmetric group gives

$$\begin{aligned}
&|\{(\sigma,\tau) \in (S_{2\lambda}B_n)^2 \mid \text{coset type}(\sigma^{-1}\tau) = \rho\}| \\
&= |\{(\sigma, r, \tau, s) \in (S_{2\lambda} \times R)^2 \mid (\sigma r)^{-1}\tau s \in B_n x_\rho B_n\}| \\
&= |\{(\sigma, r, \tau, s) \in (S_{2\lambda} \times R)^2 \mid r^{-1}\sigma^{-1}\tau s \in B_n x_\rho B_n\}| \\
&= |R|^2 |\{(\sigma,\tau) \in S_{2\lambda}^2 \mid \sigma^{-1}\tau \in B_n x_\rho B_n\}| && (\text{as } r, s \in R \subseteq B_n) \\
&= |S_{2\lambda}|\,|R|^2 |\{\sigma \in S_{2\lambda} \mid \sigma \in B_n x_\rho B_n\}| && (\text{as } S_{2\lambda} \text{ is a group}) \\
&= |S_{2\lambda}||R|^2 |S_{2\lambda} \cap B_n x_\rho B_n| \\
&= \frac{|A|^2|B_n|^2}{|S_{2\lambda}|}|S_{2\lambda} \cap B_n x_\rho B_n| \\
&= \frac{|A||B_n|^2}{|B_\lambda|}|S_{2\lambda} \cap B_n x_\rho B_n|. && (\text{as } |A| = \frac{|S_{2\lambda}|}{|B_\lambda|})
\end{aligned}$$

Dividing by $|B_n|^2$ gives the number of pairs in the scheme $S_{2n}/B_n$ and dividing by $|A|$ gives the inner distribution. □

### 6.2.2 *Characterisation of Designs*

Before we proceed to give a characterisation of designs in the association scheme of perfect matchings, let us present the characterisation for the symmetric group in a different way that motivates our computations with the zonal spherical functions.

We find from Frobenius reciprocity and Equation (4.1) that for $\lambda, \mu \vdash n$

$$\langle \xi^\mu, \chi^\lambda \rangle = \langle 1 \uparrow_{S_\mu}^{S_n}, \chi^\lambda \rangle = \langle 1_{S_\mu}, \chi^\lambda \downarrow_{S_\mu}^{S_n} \rangle = \frac{1}{|S_\mu|} \sum_{\sigma \in S_\mu} \chi^\lambda(\sigma) = \frac{1}{|S_\mu| \deg(\chi^\lambda)} a'_\lambda.$$

So computing the inner product of the induced character with the irreducible character $\chi^\lambda$ is essentially equivalent to computing the entry $a'_\lambda$ of the dual distribution of a Young subgroup of shape $\mu$.

We can think of a Young subgroup $Y$ of shape $\lambda$ as an antidesign for some subset $T$. Then, a $\lambda$-transitive set $D$ is a $T$-design. Hence, if we can figure out which irreducible characters are constituents of the induction of the trivial character $1_Y$, we have computed the set $T$.

Now we replace the irreducible characters with the zonal symmetric functions and the induced character with something that behaves similarly. Let

$$\theta^\lambda = e_{B_n} * 1_{2\lambda} * e_{B_n} \in \mathbb{C}[B_n \backslash S_{2n} / B_n]$$

where $1_{2\lambda}$ is the trivial character of $S_{2\lambda}$. In $\mathbb{C}[S_{2n}]$, $1_{2\lambda}$ is the characteristic vector of the subgroup $S_{2\lambda}$. Explicitly, we have

$$\theta^\lambda(\sigma) = (e_{B_n} * 1_{2\lambda} * e_{B_n})(\sigma) = \frac{1}{|B_n|^2} \sum_{a,b \in B_n} 1_{2\lambda}(a\sigma b).$$

We can decompose $\theta^\lambda$ in terms of the orthogonal basis of $\mathbb{C}[B_n \backslash S_{2n} / B_n]$ given by the zonal spherical functions $\omega^\mu$. Let $x_\rho \in S_{2n}$ be any element of coset type $\rho$, recall that $\omega^\mu$ and $\theta^\lambda$ are constant on the double cosets and calculate

$$\begin{aligned}
\langle \omega^\mu, \theta^\lambda \rangle &= \sum_{\sigma \in S_{2n}} \omega^\mu(\sigma) \overline{\theta^\lambda(\sigma)} = \sum_{\rho \vdash n} |B_n x_\rho B_n| \omega^\mu_\rho \theta^\lambda(x_\rho) \\
&= \frac{1}{|B_n|^2} \sum_{\rho \vdash n} |B_n x_\rho B_n| \, \omega^\mu_\rho \sum_{a,b \in B_n} 1_{2\lambda}(a x_\rho b).
\end{aligned}$$

So it remains to calculate the inner sum. We find

$$\sum_{a,b \in B_n} 1_{2\lambda}(a x_\rho b) = \sum_{a \in B_n} \sum_{b \in B_n} 1_{2\lambda}(a x_\rho b) = \sum_{a \in B_n} |S_{2\lambda} \cap a x_\rho B_n|.$$

We have $a x_\rho B_n = b x_\rho B_n \iff a^{-1} b \in x_\rho B_n x_\rho^{-1}$. Since $a, b \in B_n$, we find that $a x_\rho$ and $b x_\rho$ are representatives of the same coset if and only if $a^{-1} b \in B_n \cap x_\rho B_n x_\rho^{-1}$, equivalently $a(B_n \cap x_\rho B_n x_\rho^{-1}) = b(B_n \cap x_\rho B_n x_\rho^{-1})$. Let $R \subseteq B_n$ be a system of representatives of the left cosets of $B_n / (B_n \cap x_\rho B_n x_\rho^{-1})$. By the argument above, $R$ is also a system of representatives for the left cosets $r x_\rho B_n$ in $B_n x_\rho B_n$. Then we find

$$\begin{aligned}
\sum_{a \in B_n} |S_{2\lambda} \cap a x_\rho B_n| &= \sum_{r \in R} \sum_{\sigma \in B_n \cap x_\rho B_n x_\rho^{-1}} |S_{2\lambda} \cap r\sigma x_\rho B_n| \\
&= \sum_{r \in R} \sum_{\sigma \in B_n \cap x_\rho B_n x_\rho^{-1}} |S_{2\lambda} \cap r x_\rho B_n| \qquad \text{as } \sigma x_\rho B_n = x_\rho B_n \\
&= \sum_{r \in R} |B_n \cap x_\rho B_n x_\rho^{-1}| \, |S_{2\lambda} \cap r x_\rho B_n| \\
&= |B_n \cap x_\rho B_n x_\rho^{-1}| \sum_{r \in R} |S_{2\lambda} \cap r x_\rho B_n| \\
&= |B_n \cap x_\rho B_n x_\rho^{-1}| \, |S_{2\lambda} \cap B_n x_\rho B_n|.
\end{aligned}$$

We have that (see [45, Ch. VII (2.3)])

$$|B_n x_\rho B_n| = \frac{|B_n|^2}{|B_n \cap x_\rho B_n x_\rho^{-1}|}.$$

Putting it all together, we obtain

$$\begin{aligned}
\langle \omega^\mu, \theta^\lambda \rangle &= \frac{1}{|B_n|^2} \sum_{\rho \vdash n} |B_n x_\rho B_n| \, \omega_\rho^\mu \sum_{a \in B_n} |S_{2\lambda} \cap a x_\rho B_n| \\
&= \frac{1}{|B_n|^2} \sum_{\rho \vdash n} |B_n x_\rho B_n| \, \omega_\rho^\mu \, |B_n \cap x_\rho B_n x_\rho^{-1}| \, |S_{2\lambda} \cap B_n x_\rho B_n| \\
&= \sum_{\rho \vdash n} \omega_\rho^\mu \, |S_{2\lambda} \cap B_n x_\rho B_n| = 2^n \lambda! \sum_{\rho \vdash n} \omega_\rho^\mu a_\rho(A) \quad \text{(Lemma 6.2.7)} \\
&= \frac{2^n \lambda!}{\chi^{2\mu}(1)} \chi^{2\mu}(1) \sum_{\rho \vdash n} \omega_\rho^\mu a_\rho(A) = \frac{2^n \lambda!}{\chi^{2\mu}(1)} a'_\mu(A).
\end{aligned}$$

Hence, $\langle \omega^\mu, \theta^\lambda \rangle$ gives the dual distribution of $A$ up to a non-zero constant.

Indeed, $\langle \omega^\mu, \theta^\lambda \rangle$ is a positive multiple of the coefficient $u_{\mu,\lambda}$ of the monomial symmetric function $m_\lambda$ in the expansion of the zonal polynomial $Z_\mu$ (see the proof of [45, Ch. VII (2.22)]). This coefficient is 0 unless $\mu \trianglerighteq \lambda$.

Furthermore, we find from [45, Ch. VII (2.23)] that $Z_\mu$ is the *Jack symmetric function* with parameter $\alpha = 2$. By [45, Ch. VI (10.13), (10.15)], $u_{\mu,\lambda} > 0$ whenever $\mu \trianglerighteq \lambda$. It follows that

$$\langle \omega^\mu, \theta^\lambda \rangle \neq 0 \iff \mu \trianglerighteq \lambda.$$

We can now prove Theorem 6.2.4.

*Proof of Theorem 6.2.4.* A $\lambda$-factorisation $D$ is a $T$-design where $T$ is the set of partitions such that the set $A$ of left cosets of $B_n$ with representatives in $S_{2\lambda}B_n$ is a $T$-antidesign. By the calculation above, the dual distribution of $A$ is $a'_\mu(A) = C\langle \omega^\mu, \theta^\lambda \rangle$ for some non-zero constant $C$. Since $\langle \omega^\mu, \theta^\lambda \rangle \neq 0$ if and only if $\mu \trianglerighteq \lambda$, the result follows. $\square$

## 6.3 COMPARISON OF DESIGNS

By characterising $\lambda$-factorisations in terms of the dual distribution, we can compare $\lambda$-factorisations of different types.

**Theorem 6.3.1.** *Let $\lambda, \mu \vdash n$ and $\varnothing \neq D \subseteq S_{2n}/B_n$ be a $\lambda$-factorisation. If $\mu \trianglerighteq \lambda$, then $D$ is also a $\mu$-factorisation.*

*Proof.* Follows immediately from Theorem 6.2.4 and the fact that the dominance order $\trianglelefteq$ is transitive. $\square$

Notice the similarity of Theorem 6.3.1 and Theorem 4.3.7.

**Remark 6.3.2.** Notice that the converse of Theorem 6.3.1 does not necessarily hold if $\lambda$ is not 'small' enough. Consider $n = 6$ and suppose that $D$ is a $(321)$-factorisation. Then $D$ is also a $\mu$-factorisation for $\mu \in \{(33), (411), (42), (51)\}$. Now $D$ is a $\lambda$-factorisation where $\lambda = (411)$, but we also have that $D$ is a $\mu$-factorisation where $\mu = (33)$; but $\lambda$ and $\mu$ are incomparable in the dominance order.

Theorem 6.3.1 gives divisibility constraints on the indices reminiscent of the arithmetic conditions on the parameters of block designs.

**Theorem 6.3.3.** *Let $\lambda, \mu \vdash n$ with $\mu \trianglerighteq \lambda$ and $\varnothing \neq D \subseteq S_{2n}/B_n$ be a $\lambda$-factorisation of index $c_\lambda$. Then $D$ is a $\mu$-factorisation of index $c_\mu$ where*

$$c_\mu = c_\lambda \frac{(2\mu_1 - 1)!!(2\mu_2 - 1)!! \dots}{(2\lambda_1 - 1)!!(2\lambda_2 - 1)!! \dots}.$$

*Hence, $(2\lambda_1 - 1)!!(2\lambda_2 - 1)!! \dots$ divides $c_\lambda(2\mu_1 - 1)!!(2\mu_2 - 1)!! \dots$ for all $\mu \trianglerighteq \lambda$.*

*Proof.* We count the pairs $(m, P)$ of matchings $m \in D$ that refine a set partition $P$ of shape $2\lambda$ in two ways. First, we count the number of set partitions of $\{1, \dots, 2n\}$ of shape $2\lambda$, and then we count the number of set partitions that one matching $m$ refines.

Denote by $m_i$ the number of parts of $\lambda$ of size $i$. Then the number of set partitions of shape $2\lambda$ is

$$\frac{(2n)!}{\prod_{i \geqslant 1, m_i > 0} (2i)!^{m_i} m_i!}$$

because the parts of a set partition of shape $2\lambda$ have even size and the parts of size $2i$ are stabilised by the wreath product $S_{2i} \wr S_{m_i}$.

Now consider any perfect matching $m$. Counting the number of set partitions that $m$ refines comes down to counting in how many ways the $n$ edges of $m$ can be grouped into parts according to $\lambda$. We find that there are

$$\frac{n!}{\prod\limits_{i\geqslant 1, m_i>0} i!^{m_i} m_i!}$$

set partitions of shape $2\lambda$ that $m$ refines. Hence, double counting the pairs $(m, P)$ of matchings $m \in D$ that refine a set partition $P$ of shape $2\lambda$, we obtain

$$\begin{aligned} & c_\lambda \frac{(2n)!}{\prod\limits_{i\geqslant 1, m_i>0} (2i)!^{m_i} m_i!} = |D| \frac{n!}{\prod\limits_{i\geqslant 1, m_i>0} i!^{m_i} m_i!} \\ \iff & |D| = c_\lambda \frac{(2n)!}{n!} \prod_{i\geqslant 1, m_i>0} \left(\frac{i!}{(2i)!}\right)^{m_i}. \end{aligned}$$

Now $(2n)!/n! = 2^n(2n-1)!!$ and $i!/(2i)! = 1/(2^i(2i-1)!!)$. In the product, we find the factors $2^{im_i}$. Because of $\sum_{i\geqslant 1} im_i = n$, the powers of 2 cancel and we can rewrite the product as

$$|D| = c_\lambda (2n-1)!! \prod_{i=1}^{l(\lambda)} \frac{1}{(2\lambda_i - 1)!!}.$$

By Theorem 6.3.1, we find that $D$ is also a $\mu$-factorisation, so we can also compute the size of $D$ in terms of $\mu$. Equating both counts gives

$$\begin{aligned} & c_\lambda (2n-1)!! \prod_{i=1}^{l(\lambda)} \frac{1}{(2\lambda_i - 1)!!} = c_\mu (2n-1)!! \prod_{i=1}^{l(\mu)} \frac{1}{(2\mu_i - 1)!!} \\ \iff & c_\mu = c_\lambda \frac{(2\mu_1 - 1)!!(2\mu_2 - 1)!! \ldots}{(2\lambda_1 - 1)!!(2\lambda_2 - 1)!! \ldots}. \end{aligned}$$

Since $c_\mu$ is an integer, the result follows. □

The formula in Theorem 6.3.3 also reveals a simple way to obtain the size of a $\lambda$-factorisation $D$ of index $c_\lambda$ (that is, use $\mu = (n)$):

$$|D| = c_\lambda \frac{(2n-1)!!}{(2\lambda_1 - 1)!!(2\lambda_2 - 1)!! \ldots}. \tag{6.2}$$

The following is a simple yet powerful special case of Theorem 6.3.3.

**Corollary 6.3.4.** *Let $\lambda \vdash n$, $\lambda \neq (n)$, and let $D \neq \varnothing$ be a $\lambda$-factorisation of index $c_\lambda$. If $k, l$ with $k \leqslant l$ are distinct parts of $\lambda$ (which may be the same if $\lambda$ has multiple parts of the same size), then $2k-1$ divides $(2l+1)c_\lambda$. In particular, if $D$ is of index $1$, then $2k-1$ divides $2l+1$.*

*Proof.* Write $\lambda = \widetilde{\lambda} \cup (l,k)$ and let $\mu = \widetilde{\lambda} \cup (l+1, k-1)$. Then we have $\lambda \trianglelefteq \mu$. Now use Theorem 6.3.3 to find that

$$c_\mu = c_\lambda \frac{(2(k-1)-1)!!(2(l+1)-1)!!}{(2k-1)!!(2l-1)!!} = c_\lambda \frac{2l+1}{2k-1}. \qquad \square$$

This severely restricts the possible partitions $\lambda$ such that a $\lambda$-factorisation of index 1 exists. Applying Theorem 6.3.3 to partitions with two parts, we get the following:

**Corollary 6.3.5.** *Let $t \leqslant n/2$ and $\varnothing \neq D \subseteq S_{2n}/B_n$ be an $(n-t,t)$-factorisation of index $c_t$. Then $D$ is an $(n-t+1, t-1)$ factorisation of index $c_{t-1}$ where*

$$c_{t-1} = c_t \frac{2n-2t+1}{2t-1} = c_t \frac{2n}{2t-1} - c_t.$$

*Proof.* Use Theorem 6.3.3 with $\lambda = (n-t,t)$ and $\mu = (n-t+1, t-1)$, which satisfy $\mu \trianglerighteq \lambda$. $\square$

Notice that $c_{t-1}$ is an integer if and only if $2t-1 \mid c_t \cdot 2n$. This gives the following non-existence result.

**Corollary 6.3.6.** *For every $n \geqslant 4$, an $(n-2,2)$-factorisation of index coprime to 3 (in particular index 1) can only exist if $n \equiv 0 \pmod 3$.*

*Proof.* Use Corollary 6.3.5 for $t = 2$. $\square$

Note that Corollary 6.3.5 can be applied repeatedly. That is, an $(n-t,t)$-factorisation is an $(n-s,s)$-factorisation for every $s = t-1, t-2, \ldots, 1$. For example, an $(n-3,3)$-factorisation of index 1 (where $n \geqslant 6$) is an $(n-2,2)$-factorisation of index $\frac{2n-5}{5}$, so $n \equiv 0 \pmod 5$. Now an $(n-2,2)$-factorisation of index $\frac{2n-5}{5}$ is an $(n-1,1)$-factorisation of index $\frac{2n-5}{5} \cdot \frac{2n-3}{3}$, so $(2n-5)(2n-3) \equiv 0 \pmod 3$, hence $n \equiv 0, 1 \pmod 3$. So an $(n-3,3)$-factorisation of index 1 can only exist if $n \equiv 0, 10 \pmod{15}$. In general, we get the following divisibility criteria.

**Theorem 6.3.7.** *Let $t \leqslant n/2$. If an $(n-t,t)$-factorisation of index 1 exists, then*

$$(2t-1)(2t-3)\ldots(2k+1) \text{ divides } (2n-(2t-1))(2n-(2t-3))\ldots(2n-(2k+1))$$

*for every $k = t-1, t-2, \ldots, 1$. Equivalently,*

$$\frac{(2(n-k)-1)!!(2k-1)!!}{(2(n-t)-1)!!(2t-1)!!}$$

*is an integer for every $k = t-1, t-2, \ldots, 1$.*

*Proof.* Apply Theorem 6.3.3 or Corollary 6.3.5 to the chain $(n-t,t) \trianglelefteq (n-t+1, t-1) \trianglelefteq \ldots \trianglelefteq (n-2,2) \trianglelefteq (n-1,1)$. $\square$

Note that Theorem 6.3.7 also holds in the case $k = 0$ but the resulting divisibility condition is a weaker version of the divisibility condition for $k = 1$.

For many combinatorial objects, there exists a process called *derivation* that builds smaller objects from bigger ones. This is also the case for $\lambda$-factorisations.

**Definition 6.3.8.** Let $\lambda \vdash n$, $\lambda \neq (n)$, and $D$ be a $\lambda$-factorisation. Let $k$ be a part of $\lambda$ and $S \subseteq \{1, \ldots, 2n\}$ be a set of size $2k$. For a perfect matching $m \in D$, we write $m \setminus S$ for the set of edges of $m$ that are disjoint from $S$. Then the set

$$D_S = \{m \setminus S \mid m \text{ refines } \{S, [2n] \setminus S\}\}$$

is called the *derivation* of $D$ at $S$.

**Example 6.3.9.** Consider an $(n-2, 1, 1)$-factorisation $D$ of index $c$ and let $e = \{2n-1, 2n\}$ be an edge of $K_{2n}$. Then we consider all matchings $m \in D$ that contain $e$ and remove the edge $e$ from all of them. Hence, the perfect matchings $m \setminus e \in D_e$ are perfect matchings of $K_{2(n-1)}$ (on the set $[2(n-1)]$).

Now let $f$ be any edge of $K_{2(n-1)}$. Since $D$ is an $(n-2, 1, 1)$-factorisation of index $c$, there are precisely $c$ matchings $m \in D$ that contain both the edge $e$ and the edge $f$. By definition, a matching $m \in D$ containing $e$ and $f$ gives rise to an edge $m \setminus e$ containing $f$. So we find that there are precisely $c$ matchings containing $f$ and $D_e$ is an $(n-1, 1)$-factorisation of index $c$.

The following theorem is immediate.

**Theorem 6.3.10.** *Let $\lambda \vdash n$, $\lambda \neq (n)$, and $D$ be a $\lambda$-factorisation of index $c$. Let $k$ be a part of $\lambda$, let $S \subseteq [2n]$ be a set of size $2k$ and write $\lambda = \widetilde{\lambda} \cup (k)$ for a partition $\widetilde{\lambda} \vdash n-k$. Then the derivation $D_S$ is a $\widetilde{\lambda}$-factorisation of index $c$.*

*Proof.* Without loss of generality, we can assume that $S = \{2(n-k)+1, \ldots, 2n\}$ (otherwise, permute the elements of $[2n]$ suitably). Consider the set $P_S$ of all set partitions of $[2n]$ of shape $2\lambda$ that contain $S$. We find

$$P_S = \{S \cup \widetilde{P} \mid \widetilde{P} \text{ is a set partition of } [2(n-k)] \text{ of shape } 2\widetilde{\lambda}\}.$$

Now let $\widetilde{P}$ be any set partition of $[2(n-k)]$ of shape $2\widetilde{\lambda}$. Then $S \cup \widetilde{P} \in P_S$ and $S \cup \widetilde{P}$ has shape $2\lambda$. Since $D$ is a $\lambda$-factorisation, there are exactly $c$ perfect matchings $m \in D$ that refine $S \cup \widetilde{P}$. All matchings $m \in D$ that refine a set partition of $P_S$ also refine the set partition $\{S, [2n] \setminus S\}$ and thus give rise to elements $m \setminus S$ of $D_S$. Hence, there are exactly $c$ perfect matchings $\widetilde{m} \in D_S$ that refine $\widetilde{P}$ and $D_S$ is a $\widetilde{\lambda}$-factorisation of index $c$. □

For two partitions $\lambda, \mu$, we write $\mu \preccurlyeq \lambda$ if every part of $\mu$ is a part of $\lambda$. Then Theorem 6.3.10 implies the following.

**Corollary 6.3.11.** *Let $\lambda \vdash n$ and $D$ be a $\lambda$-factorisation of index $c$. Then there exist $\mu$-factorisations of index $c$ for every partition $\mu \preccurlyeq \lambda$.*

*Proof.* Use derivation multiple times. □

Corollary 6.3.11 implies that if a $\lambda$-factorisation of index $c$ does not exist, then there is no $\mu$-factorisation of index $c$ where all parts of $\lambda$ turn up as parts of $\mu$. Thus, knowledge of the existence or non-existence of $\lambda$-factorisations where $\lambda \vdash n$ with small $n$ gives insight into the existence for bigger values of $n$. For example, we can generalise a result of Cameron [12, Theorem 7.6] and Mathon [48] that there is no 2-$(2, 8)$ partition system.

**Corollary 6.3.12.** *Let $n \geqslant 4$. There is no $(2, 1, \dots, 1)$-factorisation of index 1. In particular, there is no $(n-2)$-$(2, 2n)$ partition system.*

*Proof.* By Cameron [12, Theorem 7.6] and Mathon [48], there is no (211)-factorisation of index 1. Now the result follows from Corollary 6.3.11. □

Alternatively, the theory of symmetric functions can be used for a longer proof of Corollary 6.3.12. We present it here, because it also shows that there is a Krein parameter of the association scheme on perfect matchings that is guaranteed to be 0, and this is of independent interest.

**Example 6.3.13.** Let $S$ be a putative $(2, 1, \dots, 1)$-factorisation of index 1. By Theorem 6.2.4, the dual distribution of $S$ is of the form $(|S|, 0, \dots, 0, x)$ with only two non-zero entries. The coordinate of the dual distribution indexed by the trivial partition $(n)$ is equal to $|S|$ and the other non-zero entry is a positive rational number $x$ that appears in the coordinate indexed by the partition $\mu := (1, 1, \dots, 1)$.

For subsets with a dual distribution of this special form, the Krein parameter $q^{\mu}_{\mu\mu}$ carries valuable information. It was shown in [4, Corollary 1.5] that if $q^{\mu}_{\mu\mu} = 0$, then $S$ consists of precisely half of the elements of the association scheme. But for the perfect matching association scheme, this is a contradiction because there is an odd number of perfect matchings of $K_{2n}$. Hence, showing $q^{\mu}_{\mu\mu} = 0$ implies that there is no $(2, 1, \dots, 1)$-factorisation of index 1.

We now use the formula given in [5, Theorem 3.6 and 3.5(i)] to calculate the Krein parameter. The valency $k_\rho$ of the perfect matching scheme is the size of the $\rho$-sphere $\Omega_\rho$. This is the size of the double coset $B_n x_\rho B_n$ of type $\rho$. By [45, Ch. VII (2.3)], we find that the size is given by the formula $|B_n|^2 z_{2\rho}^{-1}$ where $z_{2\rho}$ is a non-zero number whose exact value need not concern us. The eigenvalue $P_\mu(\rho)$ of the perfect matching scheme is given by the formula $P_\mu(\rho) = k_\rho \omega^{\mu}_{\rho}$ (which follows from a general result about Gelfand pairs, see [27, Lemma 13.8.3]).

We will use the relation $\equiv$ to denote 'up to a non-zero scalar'. Using the formula for the value of the zonal spherical function $\omega_\rho^{(1,\dots,1)}$ given in [45, Ch. VII Ex. 2(b)], we find

$$\begin{aligned} q_{\mu\mu}^{\mu} &\equiv \sum_{\rho\vdash n} \frac{1}{k_\rho^2} \left(P_\mu(\rho)\right)^3 = \sum_{\rho\vdash n} k_\rho \left(\omega_\rho^{\mu}\right)^3 \equiv \sum_{\rho\vdash n} z_{2\rho}^{-1}\omega_\rho^{(1,\dots,1)} \cdot \left(\omega_\rho^{(1,\dots,1)}\right)^2 \\ &= \sum_{\rho\vdash n} z_{2\rho}^{-1}\omega_\rho^{(1,\dots,1)} \cdot \left(\frac{(-1)^{n-l(\rho)}}{2^{n-l(\rho)}}\right)^2 = \sum_{\rho\vdash n} z_{2\rho}^{-1}\omega_\rho^{(1,\dots,1)} \cdot \frac{1}{2^{2n-2l(\rho)}} \\ &= 4^{-n} \sum_{\rho\vdash n} z_{2\rho}^{-1}\omega_\rho^{(1,\dots,1)} \cdot 4^{l(\rho)} \\ &\equiv c_{(1,\dots,1)}(4). \end{aligned}$$

The last step follows from [45, Ch. VII Ex. 2(c)]. Here,

$$c_\lambda(X) = \prod_{(i,j)\in\lambda} (X - i + 2j - 1)$$

is the *content polynomial* (see [45, Ch. VII (2.24)]). For the partition $\mu = (1,\dots,1)$, we find

$$c_{(1,\dots,1)}(X) = \prod_{i=1}^{n}(X - i + 1).$$

Clearly, we have $c_{(1,\dots,1)}(4) = 0$ for all $n \geqslant 5$. Hence $q_{\mu\mu}^{\mu} = 0$ for $\mu = (1,\dots,1)$.

Table 6.1 gives explicit constraints that follow from Theorem 6.3.7 and Corollary 6.3.11 for $\lambda$-factorisations of index 1 where $\lambda$ has few and small parts.

**Remark 6.3.14.** A near-perfect matching is a matching in a graph where all but one vertex are matched, and the graph must have an odd number of vertices. We can define $\lambda$-factorisations for a complete graph on an odd number of vertices, say $2n-1$, by considering only the partitions $\lambda \vdash 2n-1$ that have exactly one odd part. Every such partition extends uniquely to a partition $\widehat{\lambda} \vdash 2n$ that only has even parts. Likewise, every near-perfect matching of $K_{2n-1}$ extends uniquely to a perfect matching of $K_{2n}$. Conversely, deleting the element $2n$ of a set partition of $\{1,\dots,2n\}$ with only even parts (resp. a perfect matching) leaves a set partition with exactly one odd part (resp. a near-perfect matching). Thus, there is a bijection between near-perfect matchings of $K_{2n-1}$ and perfect matchings of $K_{2n}$ and the corresponding $\lambda$-factorisations are essentially the same structures.

We could use similar techniques from the theory of association schemes to study near-perfect matchings of $K_{2n-1}$. Indeed, we obtain a natural association scheme from the Gelfand pair $(S_{2n-1}, B_{n-1})$ where $B_{n-1}$ is the centraliser of $(1\,2)(3\,4)\cdots(2n-3\;2n-2)$. See [41, Section 11] for more.

| $\lambda$ | Constraints |
|---|---|
| $(n-t, 1, \ldots, 1)$ | $t \neq n-2$ |
| $(n-1, 1)$ | none, always exists |
| $(n-2, 2)$ | $n \equiv 0 \pmod 3$ |
| $(n-3, 3)$ | $n \equiv 0, 10 \pmod{15}$ |
| $(n-3, 2, 1)$ | $n \equiv 1 \pmod 3$ |
| $(n-4, 4)$ | $n \equiv 0, 21 \pmod{35}$ |
| $(n-4, 3, 1)$ | $n \equiv 1, 11 \pmod{15}$ |
| $(n-4, 2, 2)$ | does not exist |
| $(n-4, 2, 1, 1)$ | does not exist |
| $(n-5, 5)$ | $n \equiv 0, 36, 126, 162, 225, 252 \pmod{315}$ |

Table 6.1: Criteria for the existence of $\lambda$-factorisations of index 1 for some $\lambda$.

## 6.4 EXAMPLES FOR SMALL $n$

In this section, we give an overview of the existence of $\lambda$-factorisations of index 1 for small values of $n$.

The first case is $n = 3$. The partitions of 3 are (3), (21) and (111), with the only non-trivial partition being $\lambda = (21)$. An $(n-1, 1)$-factorisation of index 1 is a 1-factorisation (see Example 6.2.2), and so exists. For $n = 4$, we have $\lambda \in \{(1111), (211), (22), (31), (4)\}$, where again (4) and (1111) are the trivial partitions that are not interesting. The case $\lambda = (31)$ corresponds to 1-factorisations so it is also not interesting. Corollary 6.3.4 shows that there is no example for $\lambda = (22)$. It was shown by Cameron [12, Theorem 7.6] and Mathon [48] that there is no example for $\lambda = (211)$.

For $n = 5$, the partitions are totally ordered in the dominance order: $(11111) \trianglelefteq (2111) \trianglelefteq (221) \trianglelefteq (311) \trianglelefteq (32) \trianglelefteq (41) \trianglelefteq (5)$. By Corollary 6.3.4, a $\lambda$-factorisation of index 1 does not exist for $\lambda \in \{(221), (32)\}$. By Corollary 6.3.11, a (2111)-factorisation of index 1 does not exist. Again, a (41)-factorisation of index 1 is a 1-factorisation (and so exists), and a (311)-factorisation is a hyperfactorisation, which exists in this case (see Example 6.2.3).

Now suppose $n = 6$. We can immediately rule out $\lambda$-factorisations of index 1 for $\lambda \in \{(2211), (222), (321), (33)\}$ with a simple application of Corollary 6.3.4. Corollary 6.3.12 rules out $\lambda = (21111)$. So the only non-trivial partitions of 6 that remain are (3111), (411), and (42). A (411)-factorisation of index 1 is equivalent to an abstract hyperoval of order 10. It was shown in [38] that such an object does not exist, via an exhaustive computer search. There is an example of a (42)-factorisation of index 1 which is explored in Example 6.4.1. To show that (3111)-factorisations of index 1 do not exist, we use the fact (Theorem 6.3.10) that the derivation of a putative example would yield a hyperfactorisation of $K_{10}$, and we know there are only two examples (up to isomorphism). So we construct the two examples, extend them to sets of perfect matchings of $K_{12}$ (by adjoining $\{11, 12\}$ to each matching), and then we use a constraint satisfaction solver such as *Gurobi* [28] to show that there is no (3111)-factorisation whose derivation is the given hyperfactorisation. The GAP code is attached as an appendix (see a.2), where we make use of the package *Gurobify* [39].

| $\lambda$ | Exists? |
|---|---|
| (21111) | No, Corollary 6.3.12 |
| (2211) | No, Corollary 6.3.4 |
| (222) | No, Corollary 6.3.4 |
| (3111) | No (by computer) |
| (321) | No, Corollary 6.3.4 |
| (33) | No, Corollary 6.3.4, Theorem 6.3.7 |
| (411) | No, [38] |
| (42) | Yes |
| (51) | Yes |

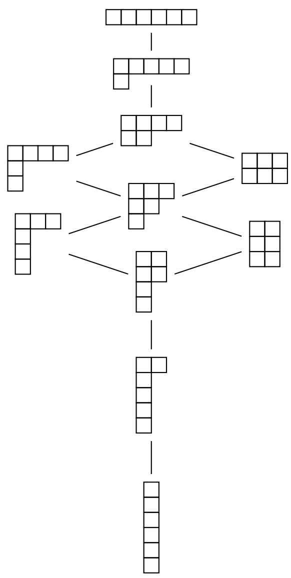


Table 6.2: The case $n = 6$. On the right is the dominance Hasse diagram of the partitions of 6. On the left is the existence data for $\lambda$-factorisations of index 1.

**Example 6.4.1.** Here, we give an example of a (42)-factorisation. First, we let $\mathrm{AGL}(1, 11)$ be the stabiliser of $\infty$ under the action of $\mathrm{PGL}(2, 11)$ on the projective line $\mathrm{PG}(1, 11)$. Let $\mathbb{F}_{11}^{\square}$ be the set of non-zero squares in the finite field $\mathbb{F}_{11}$ of order 11. Consider the following two perfect matchings on the 12 points of the projective line $\mathrm{PG}(1, 11) = \mathbb{F}_{11} \cup \{\infty\}$:

$$\{\{0, \infty\}\} \cup \{\{x, -x\} \mid x \in \mathbb{F}_{11}^{\square}\}, \quad \{\{0, \infty\}\} \cup \{\{x, 7x\} \mid x \in \mathbb{F}_{11}^{\square}\}$$

Then the orbits of $\mathrm{AGL}(1, 11)$ on these perfect matchings are orbits of size 11 and 22, respectively. Their union forms a (42)-factorisation of index 1. Note that this example is very much related to the Steiner system with parameters 5-$(12, 6, 1)$ – the *small Witt design*. If we take the subset $\{\infty\} \cup \mathbb{F}_{11}^{\square}$ of the projective line $\mathrm{PG}(1, 11)$ and its orbit under $\mathrm{PSL}(2, 11)$, then we have 132 subsets of size 6 such that any 5 points lie in a unique block.

We failed to generalise this example to more than this one example, despite the large amount of symmetry that this example possesses. It would be interesting to find $\lambda$-factorisations admitting $\mathrm{AGL}(1, q)$ as automorphisms, where $2n = q + 1$.

For $n = 7$, we can again apply Corollary 6.3.4 and Corollary 6.3.12 to show that most of the $\lambda$-factorisations of index 1 do not exist. The only non-trivial partitions to consider are $(421)$ and $(511)$. A $(511)$-factorisation of index 1 is a hyperfactorisation of $K_{14}$ of index 1. To our knowledge, it is unknown whether such a hyperfactorisation exists. If it were to exist, then it would also give rise to a 5-design on 14 points with block size 6 and index 15 (see [8, p. 10]). We emphasise that this 5-design would have repeated blocks as the complete design on 14 points with block size 6 is a 5-design of index 9. There is a 5-design on 14 points with block size 6 and index 3 due to Brouwer [10] that does not have repeated blocks. The 5-design obtained from a hypothetical $(511)$-factorisation of index 1 could be a 5-fold copy of the design by Brouwer. Finally, we could not determine by computer or otherwise whether a $(421)$-factorisation of index 1 exists or not.

For $n = 8$, after applying Corollary 6.3.4 and Corollary 6.3.12, we are left with non-trivial $\lambda$-factorisations of index 1 with $\lambda \in \{(5111), (611)\}$. Again, we do not know whether these partitions have examples or not. One can see a pattern emerging here where the possible valid partitions have relatively small length. This is because many $\lambda$-factorisation of index 1 for small $n$ do not exist. By Corollary 6.3.11, no $\lambda$-factorisation of index 1 can exist if a $\widetilde{\lambda}$-factorisation of index 1 for some $\widetilde{\lambda} \preccurlyeq \lambda$ does not exist. For example, by Corollary 6.3.12, no $\lambda$-factorisation of index 1 can have $(211)$ among its parts (for any $n$).

Finally, for $n = 9$, we have an example of a hyperfactorisation of index 1 of $K_{18}$ from Cameron's construction, and so the case $\lambda = (711)$ turns out to have an example. So the remaining partitions where we do not have a resolution to the existence question are $(51111), (6111), (72)$.

## 6.5 CODES

In this section, we briefly look at codes of perfect matchings. We will see that there is a natural notion of a $d$-code in the perfect matching association scheme.

It is well-known that the permutations of $n$ elements are in bijection with perfect matchings of the complete bipartite graph $K_{n,n}$. The fixed point distance of two permutations (see Section 4.2) translates to perfect matchings nicely because common fixed points correspond to common edges of perfect matchings. We can transfer this idea to the non-bipartite case.

Let us first define a metric on perfect matchings. We can think of perfect matchings of $K_{2n}$ as (unordered) set partitions of shape $(2,\ldots,2)$. Write $\mathcal{M}_{2n}$ for the set of all perfect matchings of $K_{2n}$. For two elements $m,\widetilde{m} \in \mathcal{M}_{2n}$, we have that $m \cap \widetilde{m}$ is the set of edges of $K_{2n}$ that are contained both in $m$ and $\widetilde{m}$.

**Lemma 6.5.1.** *Let $n \in \mathbb{N}$. Then the map*

$$d : \mathcal{M}_{2n} \times \mathcal{M}_{2n} \to \{0,\ldots,n\},\ d(m,\widetilde{m}) = n - |m \cap \widetilde{m}|$$

*for every $m,\widetilde{m} \in \mathcal{M}_{2n}$ is a metric on $\mathcal{M}_{2n}$.*

*Proof.* Let $m,\widetilde{m} \in \mathcal{M}_{2n}$ be two perfect matchings. It is clear that $d \geqslant 0$, $d(m,\widetilde{m}) = 0 \iff m = \widetilde{m}$ and $d(m,\widetilde{m}) = d(\widetilde{m},m)$.

Now let $x,y,z \in \mathcal{M}_{2n}$ be perfect matchings with $d(x,z) = k$. Then $x$ and $z$ have precisely $n-k$ edges in common. We consider the sum $d(x,y)+d(y,z)$. There are $n$ edges in $y$ and each edge contributes 0, 1 or 2 to the sum (depending on whether it is contained in $x$ and $z$, exactly one of them or none of them). There are exactly $n-k$ edges contained in both $x$ and $z$, so no more than $n-k$ edges of $y$ can contribute 0 to the sum $d(x,y)+d(y,z)$. Hence, there are at least $k$ more edges in $y$, each contributing at least 1 to $d(x,y)+d(y,z)$, so

$$d(x,y)+d(y,z) \geqslant k = d(x,z)$$

as desired. □

As is customary, we call a non-empty set $C \subseteq \mathcal{M}_{2n}$ a *d-code* if $d(m,\widetilde{m}) \geqslant d$ for every distinct elements $m,\widetilde{m} \in C$. Notice that just as with permutations, there are no perfect matchings that have distance precisely 1.

Recall that we have $r(m,\widetilde{m}) = \mu$ if the disjoint union $m \cup \widetilde{m}$ consists of cycles of length $2\mu$. It is clear that the 1s in $\mu$ correspond to common edges of $m$ and $\widetilde{m}$. Hence, we find that if $r(m,\widetilde{m}) = \mu$, then

$$d(m,\widetilde{m}) = n - \text{number of 1s in } \mu = n - (\mu'_1 - \mu'_2).$$

It follows that

$$d(m,\widetilde{m}) < d \iff n - (\mu'_1 - \mu'_2) < d \iff \mu'_1 - \mu'_2 > n - d.$$

Note that $m = \widetilde{m}$ if and only if $\mu = (1^n)$, so $\mu'_1 - \mu'_2 = n$. This immediately gives the following characterisation using the inner distribution.

**Theorem 6.5.2.** *Let $n \in \mathbb{N}$ and $1 \leqslant d \leqslant n$. Let $C \subseteq \mathcal{M}_{2n}$ be a non-empty set of perfect matchings with inner distribution $a$. Then*

$$C \text{ is a } d\text{-code} \iff a_\mu = 0 \text{ for every } \mu \vdash n \text{ with } n - d + 1 \leqslant \mu'_1 - \mu'_2 < n.$$

Notice that the size of a $(d-1, 1^{n-d+1})$-factorisation of index 1 is an upper bound for the size of a $d$-code. If $C \subseteq \mathcal{M}_{2n}$ is a $d$-code, then for any set of $n-d+1$ independent edges of $K_{2n}$ there can be at most 1 element in $C$ that contains all these edges (otherwise, there would be two distinct matchings at a distance of at most $d-1$). Equality holds if and only if $C$ is a $(d-1, 1^{n-d+1})$-factorisation of $K_{2n}$ of index 1. We have already seen this relation between cliques and designs in the generalised symmetric group in Theorem 5.4.5.

We summarise our observations in the following theorem.

**Theorem 6.5.3.** *Let $n \in \mathbb{N}$, let $1 \leqslant d \leqslant n$ and let $C \subseteq \mathcal{M}_{2n}$ be a $d$-code. Then*

$$|C| \leqslant \frac{(2n-1)!!}{(2d-3)!!} = (2n-1) \cdot (2n-3) \cdot \ldots \cdot (2d-1).$$

*Equality holds if and only if $C$ is a $(d-1, 1^{n-d+1})$-factorisation of index 1.*

*Proof.* The size of a $(d-1, 1^{n-d+1})$-factorisation can be obtained using Equation (6.2) with $\lambda = (d-1, 1^{n-d+1})$. □

Compare this to the upper bound of $n!/(d-1)!$ for permutation codes from Theorem 4.2.3.

Note that it follows from Corollary 6.3.12 that for $n \geqslant 4$, there is no 3-code $C \subseteq \mathcal{M}_{2n}$ that achieves equality in Theorem 6.5.3 because such a code would be a $(2, 1^{n-2})$-factorisation of index 1.

There are many questions one can ask about codes of perfect matchings. Their similarities and connections to permutation codes make a practical application likely.

## 6.6 OPEN PROBLEMS

We list some open problems for further research that the reader might be interested to work on. We expect that they are roughly ordered increasing in difficulty.

(1) Can we construct more $(n-2, 2)$-factorisations that are not also hyperfactorisations like Example 6.4.1? Is there an infinite family?

(2) Can we construct any non-trivial $(n-3, 3)$-factorisation? Since this would also yield an $(n-2, 2)$-factorisation by Theorem 6.3.1, one should investigate $(n-2, 2)$-factorisations first.

(3) Can we construct any non-trivial $(n-3,1,1,1)$-factorisation?

(4) Can we investigate similar questions for the coset algebra $S_{kn} / (S_k \wr S_n)$ of uniform set partitions? Since this is not a Gelfand pair for $k \geqslant 3$, a different approach is needed.

# a

# APPENDIX

## A.1 A sharply transitive subset of $C_2 \wr S_5$

Here, we give an explicit example of a 2-design $Y$ of index 1 (that is a sharply (311)-transitive set) in the group $C_2 \wr S_5$. It is neither a subgroup nor a coset of a subgroup. In fact, the group generated by $Y$ is the full group $C_2 \wr S_5$. We write the elements that the group acts on as $1, \overline{1}, 2, \overline{2}, 3, \overline{3}, 4, \overline{4}, 5, \overline{5}$.

We can write our set $Y$ as $Y = RH$ where $R$ is a set of 20 elements and $H$ is a set of 4 elements. For better readability, we give the generalised permutations in one line notation. That is, we write down the sequence $\big(\sigma(1), \sigma(2), \sigma(3), \sigma(4), \sigma(5)\big)$ for the permutation $\sigma$. From the definition of the hyperoctahedral group, it follows that if $\sigma(x) = y$, then $\sigma(\overline{x}) = \overline{y}$, so the one line notation contains all information about the generalised permutation.

$$
\begin{aligned}
H = &\{1\,2\,3\,4\,5,\ 1\,\overline{2}\,\overline{3}\,\overline{4}\,\overline{5},\ 1\,3\,2\,5\,4,\ 1\,\overline{3}\,\overline{2}\,\overline{5}\,\overline{4}\} \\
R = &\{1\,2\,3\,4\,5,\ 1\,4\,5\,2\,3,\ \overline{1}\,2\,\overline{3}\,5\,\overline{4},\ \overline{1}\,4\,\overline{5}\,3\,\overline{2}, \\
&\quad 2\,1\,4\,\overline{3}\,5,\ 2\,3\,5\,1\,\overline{4},\ \overline{2}\,1\,\overline{4}\,5\,3,\ \overline{2}\,3\,\overline{5}\,\overline{4}\,\overline{1}, \\
&\quad 3\,1\,5\,\overline{2}\,4,\ 3\,2\,4\,1\,\overline{5},\ \overline{3}\,1\,\overline{5}\,4\,2,\ \overline{3}\,2\,\overline{4}\,\overline{5}\,\overline{1}, \\
&\quad 4\,1\,3\,2\,\overline{5},\ 4\,2\,5\,\overline{1}\,\overline{3},\ \overline{4}\,1\,\overline{3}\,\overline{5}\,\overline{2},\ \overline{4}\,2\,\overline{5}\,\overline{3}\,1, \\
&\quad 5\,1\,2\,3\,\overline{4},\ 5\,3\,4\,\overline{1}\,\overline{2},\ \overline{5}\,1\,\overline{2}\,\overline{4}\,\overline{3},\ \overline{5}\,3\,\overline{4}\,\overline{2}\,1\}
\end{aligned}
$$

This transitive set cannot come from the construction of Proposition 5.7.1 because there is no orthogonal array of strength 2 over the alphabet $\{0, 1\}$ with 5 columns and index 1.

To see this, assume that $A$ was such an array. Without loss of generality, we can assume that the first row of $A$ is 00000 and the first two columns of $A$ are 00, 01, 10 and 11. That is

$$
A = \begin{matrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & & & \\ 1 & 0 & & & \\ 1 & 1 & & & \end{matrix} .
$$

Now notice that because of the first column, every entry in row 2 other than the first entry has to be 1 (otherwise, 00 would appear twice). For the same reason, because of the second column, every entry in row 3 other than the second entry has to be 1. But then columns 3 and 4 contain 11 twice. Hence, there cannot be such an orthogonal array.

The following GAP code can be copied into GAP to quickly construct the set $Y$. Note that the group $C_2 \wr S_5$ is represented here as a subgroup of $S_{10}$ under the identification

$$\{1,2,3,4,5\} \leftrightarrow \{1,3,5,7,9\} \text{ and } \{\bar{1},\bar{2},\bar{3},\bar{4},\bar{5}\} \leftrightarrow \{2,4,6,8,10\}.$$

Further, note that composition of permutations in GAP is left to right. That is, constructing $HR$ in GAP gives the set that we denote by $RH$.

```
H:=[ (), ( 3, 4)( 5, 6)( 7, 8)( 9,10), ( 3, 5)( 4, 6)( 7, 9)( 8,10), (
    3, 6)( 4, 5)( 7,10)( 8, 9)];;
R:=[ (), (3,7)(4,8)(5,9)(6,10), (1,2)(5,6)(7,9,8,10),
    (1,2)(3,7,5,10)(4,8,6,9), (1,3)(2,4)(5,7,6,8),
  (1,3,5,9,8,2,4,6,10,7), (1,4,2,3)(5,8,10,6,7,9),
     (1,4,6,9,2,3,5,10)(7,8), (1,5,9,7,4,2,6,10,8,3),
  (1,5,7)(2,6,8)(9,10), (1,6,9,3)(2,5,10,4), (1,6,7,10)(2,5,8,9),
     (1,7,3)(2,8,4)(9,10), (1,7,2,8)(5,9,6,10),
  (1,8,9,4,2,7,10,3)(5,6), (1,8,5,10,2,7,6,9), (1,9,8,6,4,2,10,7,5,3),
     (1,9,4,6,8)(2,10,3,5,7),
  (1,10,5,4,2,9,6,3)(7,8), (1,10,2,9)(3,5,8)(4,6,7)];;
Y:=Union(List(R,r->H*r));
```

## A.2 GAP CODE

Here, we give the GAP code that was used to rule out the existence of a (3111)-factorisation of index 1 of $K_{12}$.

```
# First, we construct the two hyperfactorisations of K_{10}
n := 5;
s2n := SymmetricGroup(2*n);
matching := List([1..n], i -> [2*i-1,2*i]);
matchings := Set(Orbit(s2n, matching, OnSetsSets));;
perm := Action(s2n, matchings, OnSetsSets);;
maxs := MaximalSubgroupClassReps(perm);;
s9 := First(maxs, t -> Size(t)=Factorial(9));
IsSymmetricGroup(s9);
maxs := MaximalSubgroupClassReps(s9);;
m := First(maxs, t -> Size(t)=432);
subgroup216 := MaximalNormalSubgroups(m)[1];
orbits := Filtered(Orbits(subgroup216), t -> Size(t)<=63);;
```

```
Sort(orbits, {x,y} -> Size(x) > Size(y));
# Both examples below are equivalent to Mathon's example
o1 := Union(orbits{[1,3]});;
o2 := Union(orbits{[2,3]});;

psl := PSL(2,8);
iso := IsomorphismGroups(SymmetricGroup(9),s9);;
psl_on_matchings := Image(iso, psl);
# This example arises from a hyperoval of PG(2,8) ...
# Cameron's example
o3 := First(Orbits(psl_on_matchings), t -> Size(t)=63);;

# Now extend each one with [11,12] to get a partial configuration
extensions := List([matchings{o1},matchings{o3}], mm ->
  List(mm, t -> Concatenation(t,[[11,12]])));;

# now change n and set up the Gurobi program
LoadPackage("gurobify");
n := 6;
s2n := SymmetricGroup(2*n);
matching := List([1..n], i -> [2*i-1, 2*i]);
matchings := Set(Orbit(s2n, matching, OnSetsSets));;
perm := Action(s2n, matchings, OnSetsSets);;
lambda := [3,1,1,1];
# one set partition of shape 2*lambda
p := []; max := 0;
for s in 2*lambda do
  Add(p, max + [1..s]);
  max := Maximum(Union(p));
od;
refines := {x,y} -> ForAll(x, t -> ForAny(y, u -> IsSubset(u,t)));
cp := Filtered(matchings, t -> refines(t,p));;
antidesign1 := SubsetToCharacteristicVector(cp, matchings);;
# These are the antidesigns, as 0,1-vectors.
# We seek a set that intersects each antidesign in precisely
# 1 element.
antidesigns := Orbit(perm, antidesign1, Permuted);;

for i in [1,2] do
  Print("Doing example ", i, "\n");
  ext := extensions[i];
  varnames := List([1..Size(matchings)], i -> Concatenation("x",
     String(i)));;
```

```
  vartypes := ListWithIdenticalEntries( Size(varnames), "Binary" );;
  model := GurobiNewModel(vartypes, varnames);;

  # constraint: meets each antidesign in 1 element
  GurobiAddMultipleConstraints(model,antidesigns,
    ListWithIdenticalEntries(Size(antidesigns),"="),
    ListWithIdenticalEntries(Size(antidesigns),1));

  # constraint: derives to known hyperfactorisation
  GurobiAddConstraint(model,
        SubsetToCharacteristicVector(ext,matchings), "=", 63);
  status := GurobiOptimiseModel(model);

  if status = 3 then
    Print("no example found\n");
  else
    Print("something else has happened\n");
  fi;
od;
```

# BIBLIOGRAPHY


[1] K. Aker and M. B. Can. "From parking functions to Gelfand pairs". In: *Proceedings of the American Mathematical Society* 140 (2012), pp. 1113–1124.

[2] R. A. Bailey. *Association Schemes: Designed Experiments, Algebra and Combinatorics*. Vol. 84. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2004. DOI: 10.1017/CBO9780511610882.

[3] J. Bamberg and L. Klawuhn. "On the association scheme of perfect matchings and their designs". In: *Algebraic Combinatorics* 9.3 (2026), pp. 789–809. DOI: 10.5802/alco.490.

[4] J. Bamberg and J. Lansdown. "Implications of vanishing Krein parameters on Delsarte designs, with applications in finite geometry". In: *Algebraic Combinatorics* 6.1 (2023), pp. 197–212. DOI: 10.5802/alco.246.

[5] E. Bannai and T. Ito. *Algebraic Combinatorics I: Association Schemes*. The Benjamin/Cummings Publishing Co., Inc., Menlo Park, CA, 1984.

[6] E. Bannai, Y. Nakata, T. Okuda, and D. Zhao. "Explicit construction of exact unitary designs". In: *Advances in Mathematics* 405 (2022), Paper No. 108457, 1–25. DOI: 10.1016/j.aim.2022.108457.

[7] I. Blake, G. Cohen, and M. Deza. "Coding with Permutations". In: *Information and Control* 43 (1979), pp. 1–19.

[8] E. Boros, D. Jungnickel, and S. Vanstone. "The existence of non-trivial hyperfactorizations of $K_{2n}$". In: *Combinatorica* 11 (1991), pp. 9–15.

[9] R. C. Bose. "On the Application of the Properties of Galois Fields to the Problem of Construction of Hyper-Graeco-Latin Squares". In: *Sankhyā: The Indian Journal of Statistics* 3 (1938), pp. 323–338. URL: http://www.jstor.org/stable/40383859.

[10] Andries Brouwer. *The t-designs with* $v < 18$. Tech. rep. ZN 76/77. Mathematisch Centrum Amsterdam, 1977.

[11] T. Brylawski. "The lattice of integer partitions". In: *Discrete Mathematics* 6.3 (1973), pp. 201–219. DOI: 10.1016/0012-365X(73)90094-0.

[12] P. J. Cameron. *Parallelisms of Complete Designs*. Cambridge University Press, 1976.

[13] P. J. Cameron. *Permutation Groups*. Cambridge University Press, 1999. DOI: 10.1017/CBO9780511623677.

[14] C.-O. Chow and T. Mansour. "Asymptotic probability distributions of some permutation statistics for the wreath product $C_r \wr S_n$". In: *Online J. Anal. Comb.* 7 (2012), pp. 1–14.

[15] W. Chu, C.J. Colbourn, and P. Dukes. "Constructions for Permutation Codes in Powerline Communications". In: *Designs, Codes and Cryptography* 32 (2004), pp. 51–64.

[16] F. De Clerck. "The pseudo-geometric and geometric (t,s,s-1)-graphs". In: *Simon Stevin* 53 (1979), pp. 301–317.

[17] B. C. Cooper. "Abstract Hyperovals, Partial Geometries, and Transitive Hyperovals". PhD thesis. Colorado State University, 2015.

[18] H. S. M. Coxeter. *Regular Polytopes*. 3rd. Dover Publications, 1973.

[19] P. Delsarte. "An algebraic approach to the association schemes of coding theory". In: *Philips Research Reports Supplements* 10 (1973), pp. 1–97.

[20] P. Delsarte. "Association schemes and $t$-designs in regular semilattices". In: *Journal of Combinatorial Theory Series A* 20 (1976), pp. 230–243. DOI: 10.1016/0097-3165(76)90017-0.

[21] P. Delsarte. "Pairs of vectors in the space of an association scheme". In: *Philips Research Reports* 32.5-6 (1977), pp. 373–411.

[22] P. Delsarte and V.I. Levenshtein. "Association schemes and coding theory". In: *IEEE Transactions on Information Theory* 44.6 (1998), pp. 2477–2504. DOI: 10.1109/18.720545.

[23] J. D. Dixon and B. Mortimer. *Permutation Groups*. Springer New York, 1996. DOI: 10.1007/978-1-4612-0731-3.

[24] A. Ernst and K.-U. Schmidt. "Transitivity in finite general linear groups". In: *Mathematische Zeitschrift* 307 (2024), pp. 1–25. DOI: 10.1007/s00209-024-03511-x.

[25] G. Faina. "The B-ovals of order $q \leqslant 8$". In: *Journal of Combinatorial Theory, Series A* 36 (1984), pp. 307–314. DOI: 10.1016/0097-3165(84)90038-4.

[26] *GAP – Groups, Algorithms, and Programming, Version 4.15.1*. The GAP Group, 2025. URL: https://www.gap-system.org.

[27] C. Godsil and K. Meagher. *Erdős-Ko-Rado Theorems: Algebraic Approaches*. Cambridge University Press, 2016.

[28] Gurobi Optimization Inc. *Gurobi Optimizer Version 8.1*. http://www.gurobi.com/.

[29] M. Hall. "Projective planes". In: *Transactions of the American Mathematical Society* 54 (1943), pp. 229–277.

[30] F. Harary. *Graph Theory*. Addison-Wesley Publishing Co., Reading, Mass.-Menlo Park, Calif.-London, 1969.

[31] D. R. Hughes and F. C. Piper. *Projective Planes*. Springer New York, 1973.

[32] J. E. Humphreys. *Reflection Groups and Coxeter Groups*. Cambridge University Press, 1990. DOI: 10.1017/CBO9780511623646.

[33] T. Ito. "Designs in a coset geometry: Delsarte theory revisited". In: *European Journal of Combinatorics* 25.2 (2004), pp. 229–238. DOI: 10.1016/S0195-6698(03)00102-1.

[34] T. P. Kirkman. "On a Problem in Combinations". In: *The Cambridge and Dublin mathematical journal* 2 (1847), pp. 191–204.

[35] L. Klawuhn and K.-U. Schmidt. *Transitivity in wreath products with symmetric groups*. Preprint, arXiv. 2024. URL: https://arxiv.org/abs/2409.20495.

[36] R. Koekoek and R. F. Swarttouw. *The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue*. Tech. rep. 98. Delft University of Technology, 1998.

[37] G. Kuperberg, S. Lovett, and R. Peled. "Probabilistic existence of regular combinatorial structures". In: *Geom. Funct. Anal.* 27.4 (2017), pp. 919–972. DOI: 10.1007/s00039-017-0416-9.

[38] C. W. H. Lam, L. Thiel, S. Swiercz, and J. McKay. "The nonexistence of ovals in a projective plane of order 10". In: *Discrete Mathematics* 45 (1983), pp. 319–321. DOI: 10.1016/0012-365X(83)90049-3.

[39] J. Lansdown. "Gurobify – A GAP interface to Gurobi Optimizer, Version 2.0.0". In: (2021). DOI: 10.5281/zenodo.5025558.

[40] R.A Liebler and M.R Vitale. "Ordering the Partition Characters of the Symmetric Group". In: *Journal of Algebra* 25 (1973), pp. 487–489. DOI: 10.1016/0021-8693(73)90095-1.

[41] N. Lindzey. *Intersecting families of perfect matchings*. Preprint, arXiv. 2018. URL: https://arxiv.org/abs/1811.06160.

[42] D. Livingstone and A. Wagner. "Transitivity of finite permutation groups on unordered sets". In: *Mathematische Zeitschrift* 90 (1965), pp. 393–403. DOI: 10.1007/BF01112361.

[43] É. Lucas. *Récréations mathématiques*. Gauthier-Villars, 1883.

[44] F. J. MacWilliams and N. J. A. Sloane. *The Theory of Error-Correcting Codes*. North-Holland Publishing Company, 1977.

[45] I. G. Macdonald. *Symmetric functions and Hall polynomials*. 2nd. Oxford University Press, 1995. DOI: 10.1093/oso/9780198534891.001.0001.

[46] W. J. Martin and B. E. Sagan. "A new notion of transitivity for groups and sets of permutations". In: *Journal of the London Mathematical Society (2)* 73.1 (2006), pp. 1–13. DOI: 10.1112/S0024610705022441.

[47] W. J. Martin and H. Tanaka. "Commutative association schemes". In: *European Journal of Combinatorics* 30 (2009), pp. 1497–1525. DOI: 10.1016/j.ejc.2008.11.001.

[48] R. Mathon. "The partial geometries $pg(5,7,3)$". In: *Congr. Numer.* 31 (1981), pp. 129–139.

[49] S. J. Mayer. "On the characters of the Weyl group of type $C$". In: *Journal of Algebra* 33 (1975), pp. 59–67. DOI: 10.1016/0021-8693(75)90131-3.

[50] A. R. Prince. "Oval configurations of involutions in symmetric groups". In: *Discrete Mathematics* 174.1-3 (1997), pp. 277–282. DOI: 10.1016/S0012-365X(96)00339-1.

[51] I.A. Pushkarev. "On the representation theory of wreath products of finite groups and symmetric groups". In: *Journal of Mathematical Sciences* 96 (1999), pp. 3590–3599. DOI: 10.1007/BF02175835.

[52] B. M. I. Rands. "An association scheme for the 1-factors of the complete graph". In: *Journal of Combinatorial Theory, Series A* 34.3 (1983), pp. 301–312. DOI: 10.1016/0097-3165(83)90064-X.

[53] C. Roos. "On antidesigns and designs in an association scheme". In: *Delft Progress Report* 7.2 (1982), pp. 98–109.

[54] B. E. Sagan. *The Symmetric Group*. 2nd. Springer New York, 2001. DOI: 10.1007/978-1-4757-6804-6.

[55] A. Schrijver. *Theory of linear and integer programming*. John Wiley & Sons, Ltd., Chichester, 1986.

[56] R. Schurig. "Die Paarung der Theilnehmer eines Turniers". In: *Deutsche Schachzeitung* 41 (1886), pp. 134–137.

[57] J.-P. Serre. *Linear Representations of Finite Groups*. Springer New York, 1977. DOI: 10.1007/978-1-4684-9458-7.

[58] D. Stanton. "$t$-designs in classical association schemes". In: *Graphs and Combinatorics* 2 (1986), pp. 283–286. DOI: 10.1007/BF01788103.

[59] J. Steiner. "Combinatorische Aufgabe". In: *Journal für die reine und angewandte Mathematik (Crelle's Journal)* 45 (1853), pp. 181–182.

[60] D. R. Stinson. "Designs constructed from maximal arcs". In: *Discrete Mathematics* 97.1-3 (1991), pp. 387–393. DOI: 10.1016/0012-365X(91)90453-9.

[61] D. R. Stinson. *Combinatorial Designs*. Springer New York, 2004.

[62] H. Tarnanen. "Upper bounds on permutation codes via linear programming". In: *European Journal of Combinatorics* 20 (1999), pp. 101–114. DOI: 10.1006/eujc.1998.0272.

[63] J. G. Thompson. *Fixed point free involutions and finite projective planes*. Finite simple groups II, Proc. Lond. Math. Soc. Res. Symp., Univ. Durham, pp. 321–335, 1980.

[64] R. J. Vanderbei. *Linear Programming – Foundations and Extensions*. 5th. Springer Cham, 2020. DOI: 10.1007/978-3-030-39415-8.